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Total positivity of recursive matrices

Xi Chen, Huyile Liang, Yi Wang *

School of Mathematical Sciences, Dalian University of Technology, Dalian 116024, PR China

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ABSTRACT

Let $A = [a_{n,k}]_{n,k\geq 0}$ be an infinite lower triangular matrix defined by the recurrence

$$a_{0,0} = 1, \qquad a_{n+1,k} = r_k a_{n,k-1} + s_k a_{n,k} + t_{k+1} a_{n,k+1},$$

where $a_{n,k} = 0$ unless $n \geq k \geq 0$ and r_k, s_k, t_k are all nonnegative. Many well-known combinatorial triangles are such matrices, including the Pascal triangle, the Stirling triangle (of the second kind), the Bell triangle, the Catalan triangles of Aigner and Shapiro. We present some sufficient conditions such that the recursive matrix A is totally positive. As applications we give the total positivity of the above mentioned combinatorial triangles in a unified approach.

1. Introduction

Let $A = [a_{n,k}]_{n,k\geq 0}$ be an infinite matrix. It is called *totally positive of order* r (or shortly, TP_r), if its minors of all orders $\leq r$ are nonnegative. It is called TP if its minors of all orders are nonnegative. Let $(a_n)_{n\geq 0}$ be an infinite sequence of nonnegative

* Corresponding author.
E-mail addresses: chenximath@hotmail.com (X. Chen), lianghuyile@hotmail.com (H. Liang), wangyi@dlut.edu.cn (Y. Wang).

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numbers. It is called a *Pólya frequency sequence of order r* (or shortly, a *PF*$_r$ sequence), if its Toeplitz matrix

$$[a_{i-j}]_{i,j\geq 0} = \begin{bmatrix} a_0 & & & & \\ a_1 & a_0 & & & \\ a_2 & a_1 & a_0 & & \\ a_3 & a_2 & a_1 & a_0 & \\ \vdots & & \cdots & & \ddots \end{bmatrix}$$

is TP$_r$. It is called *PF* if its Toeplitz matrix is TP. We say that a finite sequence $a_0, a_1, \ldots, a_n$ is PF$_r$ (PF, resp.) if the corresponding infinite sequence $a_0, a_1, \ldots, a_n, 0, \ldots$ is PF$_r$ (PF, resp.). We say that a nonnegative sequence (a_n) is *log-convex* (*log-concave*, resp.) if $a_i a_{j+1} \geq a_{i+1} a_j$ ($a_i a_{j+1} \leq a_{i+1} a_j$, resp.) for $0 \leq i < j$. Clearly, the sequence (a_n) is log-concave if and only if it is PF$_2$, i.e., its Toeplitz matrix $[a_{i-j}]_{i,j\geq 0}$ is TP$_2$, and the sequence is log-convex if and only if its Hankel matrix

$$[a_{i+j}]_{i,j\geq 0} = \begin{bmatrix} a_0 & a_1 & a_2 & a_3 & \cdots \\ a_1 & a_2 & a_3 & a_4 & \cdots \\ a_2 & a_3 & a_4 & a_5 & \cdots \\ a_3 & a_4 & a_5 & a_6 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

is TP$_2$ [5].

Let $\pi = (r_k)_{k\geq 1}$, $\sigma = (s_k)_{k\geq 0}$, $\tau = (t_k)_{k\geq 1}$ be three sequences of nonnegative numbers and define an infinite lower triangular matrix

$$A := A^{\pi,\sigma,\tau} = [a_{n,k}]_{n,k\geq 0} = \begin{bmatrix} a_{0,0} & & & & \\ a_{1,0} & a_{1,1} & & & \\ a_{2,0} & a_{2,1} & a_{2,0} & & \\ a_{3,0} & a_{3,1} & a_{3,2} & a_{3,3} & \\ \vdots & & & & \ddots \end{bmatrix}$$

by the recurrence

$$a_{0,0} = 1, \qquad a_{n+1,k} = r_k a_{n,k-1} + s_k a_{n,k} + t_{k+1} a_{n,k+1}, \tag{1}$$

where $a_{n,k} = 0$ unless $n \geq k \geq 0$. Following Aigner [3], we say that $A^{\pi,\sigma,\tau}$ is the *recursive matrix* and $a_{n,0}$ are the *Catalan-like numbers* corresponding to (π, σ, τ). Such triangles arise often in combinatorics and many well-known counting coefficients are Catalan-like numbers. The following are several basic examples of recursive matrices.