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Solving piecewise linear systems in abs-normal form

Andreas Griewank *, Jens-Uwe Bernt, Manuel Radons, Tom Streubel

Department of Mathematics, Humboldt-Universität zu Berlin, Germany

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ABSTRACT

With the ultimate goal of iteratively solving piecewise smooth (PS) systems, we consider the solution of piecewise linear (PL) equations. As shown in [7] PL models can be derived in the fashion of automatic or algorithmic differentiation as local approximations of PS functions with a second order error in the distance to a given reference point. The resulting PL functions are obtained quite naturally in what we call the abs-normal form, a variant of the state representation proposed by Bokhoven in his dissertation [27]. Apart from the tradition of PL modelling by electrical engineers, which dates back to the Master thesis of Thomas Stern [26] in 1956, we take into account more recent results on linear complementarity problems and semi-smooth equations originating in the optimization community [3,25,5]. We analyze simultaneously the original PL problem (OPL) in abs-normal form and a corresponding complementary system (CPL), which is closely related to the absolute value equation (AVE) studied by Mangasarian and Meyer [14] and a corresponding linear complementarity problem (LCP). We show that the CPL, like KKT conditions and other simply switched systems, cannot be open without being injective. Hence some of the intriguing PL structure described by Scholtes in [25] is lost in the transformation from OPL to CPL. To both problems one may apply Newton variants with appropriate generalized Jacobians directly computable from the abs-normal representation. Alternatively, the CPL can be solved by Bokhoven's modulus method and related fixed point iterations.

* Corresponding author.
E-mail addresses: griewank@math.hu-berlin.de (A. Griewank), berntj@math.hu-berlin.de (J.-U. Bernt), radons@math.hu-berlin.de (M. Radons), streubel@math.hu-berlin.de (T. Streubel).

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We compile the properties of the various schemes and highlight the connection to the properties of the Schur complement matrix, in particular its partial contractivity as analyzed by Rohn and Rump [23]. Numerical experiments and suitable combinations of the fixed point solvers and stabilized generalized Newton variants remain to be realized.

1. Introduction and motivation

In many applications one encounters piecewise smooth (PS) functions that can be approximated locally with second order error by piecewise linear (PL) functions. In this paper we will assume throughout that all functions are continuous and thus, in fact, Lipschitz continuous. However, an extension to piecewise linear but possibly discontinuous problems should be in the back of our minds before we settle on data structures and interfaces. Discontinuous solution operators may arise for example, if one considers least squares problems defined by piecewise linear systems of equations.

The process of piecewise linearization of a piecewise smooth function $F : \mathcal{D} \subset \mathbb{R}^n \mapsto \mathbb{R}^m$ given by an evaluation procedure was described in [7]. The key assumption is that all nonsmoothness can be cast in terms of the absolute value function $|\cdot|$. Then piecewise linearization can be achieved in the style of algorithmic differentiation [9] by simply replacing all smooth elemental functions by their tangent line or plane (in case of binary operations or special functions) and the absolute value function by itself.

In contrast to conventional notions of differentiation one does not obtain a collection of derivative vectors or matrices at a given reference point $\mathring{x}$. Rather one arrives at a procedure for evaluating an incremental PL function $\Delta F(\mathring{x}, \Delta x) : \mathcal{D} \times \mathbb{R}^n \mapsto \mathbb{R}^m$ for which

$$F(\mathring{x} + \Delta x) = F(\mathring{x}) + \Delta F(\mathring{x}, \Delta x) + O\big(\|\Delta x\|^2\big).$$

Here the error term $\|\Delta x\|^2$ is uniform on compact subsets of $\mathcal{D} - \mathring{x}$. This means that $\Delta F(\mathring{x}, \Delta x)$ is a candidate for a nonsingular uniform Newton approximation in the sense of [5], although the local homeomorphism property is by no means guaranteed.

Throughout this paper we will only be concerned with the properties of the piecewise linearized function. We will also drop the decomposition into $F(\mathring{x})$ and the increment $\Delta F(\mathring{x}, \Delta x)$ and consider a globally defined piecewise linear continuous (PL) mapping

$$F(x) \ : \ \mathbb{R}^n \mapsto \mathbb{R}^m.$$

Like for the (possibly) underlying nonsmooth mapping, our ultimate purpose is to solve certain basic numerical tasks, in particular (un)constrained optimization, equation solving, and the numerical integration of dynamical systems.