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Algebraic and invariance properties of the group of isometries

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ABSTRACT

In this paper we investigate the algebraic structure of the isometry group of several classical Banach spaces, namely, symmetric sequence spaces, $\mathcal{C}_p(\mathcal{H})$ $(1 \leq p < \infty,\ p \neq 2)$, $\mathcal{L}(\ell^p, \ell^r)$, $1 \leq p, r < \infty$, and bounded operators $\mathcal{L}(\mathcal{H})$ endowed with Chan–Li–Tui unitarily invariant (c, p)-norms. We also identify the isometrically invariant subspaces for each of theses settings.

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1. Introduction

Let X be a Banach space and $G(X)$ the group of surjective isometries of X. This is a topological group under the standard composition of operators with topology given by operator norm. We are interested in classifying $G(X)$ in algebraic terms for various Banach spaces X. For certain Banach spaces X we determine the subspaces invariant under every member of $G(X)$.

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The problem of characterizing all the surjective linear isometries supported by a Banach space goes back to the seminal work [3] due to Banach, where Banach describes as weighted composition operators, the surjective isometries on spaces of continuous functions defined on a compact metric space, see [9,10,21]. Since Banach's initial work, the group $G(X)$ has been completely characterized in terms of its action on X for many Banach spaces. For a survey of these results we refer the reader to the survey books [13,14] and the references therein. We note that it is a consequence of a result by K. Jarosz [22] that every Banach space can be equivalently renormed so that the isometry group is isomorphic to $\mathbb{S}^1$. However, we are interested in algebraic classification of $G(X)$ when X is given its standard norm. Isometries on a given Banach space, for which a characterization is known, are often described by a list of unique "symbols". As for example, for the case of $\mathcal{C}(\Omega)$, a surjective linear isometry T, on $\mathcal{C}(\Omega)$, the space of continuous functions defined on a compact metric space Ω, is of the form

$$T(f) = \lambda f \circ \varphi \quad \forall f \in \mathcal{C}(\Omega),$$

with $\lambda : \Omega \to \mathbb{S}^1$ continuous and φ a homeomorphism of Ω. We refer to λ and φ as to the symbols associated with T. This fact motivates many of the questions considered in this paper. More precisely, we start by identifying the algebraic structure of several isometry groups using the "symbol" spaces, then we determine subspaces that are invariant under the isometry group or the action of all surjective isometries.

These questions were initially motivated by two papers of Berkson and Porta on the isometry group of $H^p(D)$. In the first paper, they observe that $G(H^p(D)) \ncong \mathrm{T} \times Aut(D)$, for $1 < p < \infty$ but it is isomorphic in the case $p = 1$. They also go on to show that the only subspace invariant under $G(H^p(D))$ is all of $H^p(D)$, see [4] and [5].

In [1], Arazy characterized the isometries of the symmetric sequence spaces E and observed that $G(E) \simeq G_0 \rtimes \mathcal{P}$, the semi-direct product of G_0, the group of all diagonal multiplications by modulus 1 complex numbers, with $\mathcal{P}$, the group of all isometries defined via the action of a permutation.

Semenov and Skorik in [27] have described the algebraic structure of the isometry group of James spaces. They consider $J_{\mathcal{C}}$ (and $J_{\mathcal{R}}$) the space of all sequences of complex (and real) numbers that converge to zero and

$$\|z\|_2 = \sup_{p_1 < p_2 < \cdots < p_{2n}} \left\{ \sum_{i=1}^{n} |z_{p_{2i-1}} - z_{2^i}|^2 \right\} < \infty.$$

It is shown that the isometry group of J_c is isomorphic to the direct product $S^1 \times \mathbb{Z}_2$ and the isometry group of $J_{\mathcal{R}}$ is isomorphic to $\mathcal{Z}_2 \times \mathcal{Z}_2$.

A Banach space X is said to be extremely noncomplex if $\|Id + T^2\| = 1 + \|T^2\|$ for every $T \in B(X)$ [23]. In that paper the authors show that $G(X)$ is a discrete Boolean group whenever X is extremely noncomplex. More recently, Gardella and Thiel in [19] showed that $G(L^p(X, \mu)) \simeq Aut_*(X, \mu) \rtimes L^0(X, S^1)$ where $L^0(X, S^1)$ is the Abelian group