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Laplacian controllability classes for threshold graphs

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A B S T R A C T

Let $\mathcal{G}$ be a graph on n vertices with Laplacian matrix $\mathbf{L}$ and let $\mathbf{b}$ be a binary vector of length n. The pair $(\mathbf{L}, \mathbf{b})$ is *controllable* if the smallest $\mathbf{L}$-invariant subspace containing $\mathbf{b}$ is of dimension n. The graph $\mathcal{G}$ is called *essentially controllable* if $(\mathbf{L}, \mathbf{b})$ is controllable for every $\mathbf{b} \notin \ker(\mathbf{L})$, *completely uncontrollable* if $(\mathbf{L}, \mathbf{b})$ is uncontrollable for every $\mathbf{b}$, and *conditionally controllable* if it is neither essentially controllable nor completely uncontrollable. In this paper, we completely characterize the graph controllability classes for threshold graphs. We first observe that the class of threshold graphs contains no essentially controllable graph. We prove that a threshold graph is completely uncontrollable if and only if its Laplacian matrix has a repeated eigenvalue. In the process, we fully characterize the set of conditionally controllable threshold graphs.
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1. Introduction

Consider the single-input linear control system

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$$\dot{\mathbf{x}}(t) = \mathbf{F}\mathbf{x}(t) + \mathbf{b}u(t) \tag{1}$$

where $\mathbf{F} \in \mathbb{R}^{n\times n}$, $\mathbf{b} \in \mathbb{R}^n$, $\mathbf{x}(t) \in \mathbb{R}^n$, and $u(t) \in \mathbb{R}$. If for each $\mathbf{x}_0 \in \mathbb{R}^n$ there exists a *control signal* $u : \mathbb{R} \to \mathbb{R}$ such that the trajectory of (1) with initial condition $\mathbf{x}(0) = \mathbf{x}_0$ reaches the origin in finite time, then the pair $(\mathbf{F}, \mathbf{b})$ is called *controllable*. It is well-known that $(\mathbf{F}, \mathbf{b})$ is controllable if and only if the smallest $\mathbf{F}$-invariant subspace containing $\mathbf{b}$, denoted by $\langle \mathbf{F}; \mathbf{b}\rangle$, has full dimension n [12]. Although controllability of linear systems is a well developed subject, the problem has drawn recent interest due to applications in networked dynamical systems and distributed control. Specifically, the case where $\mathbf{F}$ is the Laplacian matrix $\mathbf{L}$ of a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, and $\mathbf{b} \in \{0, 1\}^n$ is a binary vector, has drawn a great deal of attention in recent years [20,18,11,10,17,15]. In engineering applications, the vertices $\mathcal{V}_{\mathbf{b}} := \{v_i \in \mathcal{V} \mid (\mathbf{b})_i = 1\}$ are seen as *leader* agents and influence the remaining *follower* agents $\mathcal{V}\backslash\mathcal{V}_{\mathbf{b}}$ through the control signal $u : \mathbb{R} \to \mathbb{R}$ and the connectivity of the communication network defined by the graph $\mathcal{G}$. A major problem of interest is to characterize the controllability properties of $(\mathbf{L}, \mathbf{b})$ in terms of the topological properties of $\mathcal{G}$ as $\mathbf{b}$ is allowed to vary within the set $\{0, 1\}^n$ of binary vectors. The reason for studying (1) with the Laplacian matrix is that it serves as a benchmark for studying consensus algorithms [16], and moreover, the problem is of independent interest since its characterization reveals valuable information about the eigenvectors of the Laplacian and adjacency matrices of graphs [5,4,3].

In this paper, we study the topological obstructions to controllability via the notion of *graph controllability classes*, recently introduced in [1].

Definition 1.1. Let $\mathcal{G}$ be a connected graph with Laplacian matrix $\mathbf{L}$. Then $\mathcal{G}$ is called

(i) *essentially controllable* on $\{0, 1\}^n$ if $(\mathbf{L}, \mathbf{b})$ is controllable for every $\mathbf{b} \in \{0, 1\}^n\backslash \ker(\mathbf{L})$;
(ii) *completely uncontrollable* on $\{0, 1\}^n$ if $(\mathbf{L}, \mathbf{b})$ is uncontrollable for every $\mathbf{b} \in \{0, 1\}^n$; and
(iii) *conditionally controllable* on $\{0, 1\}^n$ if it is neither essentially controllable nor completely uncontrollable on $\{0, 1\}^n$.

For each integer $n \geq 2$, let a_n be the number of asymmetric connected graphs and let e_n be the number of essentially controllable graphs, of order n. It is known that an essentially controllable graph of order larger than two must be asymmetric. On the other hand, the block graphs of Steiner triple systems generate asymmetric graphs of arbitrarily large order that are completely uncontrollable. However, it is conjectured that $\lim_{n\to\infty} e_n/a_n = 1$.

In this paper, we consider *threshold* graphs and show that the presence of a repeated eigenvalue is a necessary condition for complete uncontrollability and in the process completely classify the set of conditionally controllable threshold graphs. Threshold graphs were introduced in [2] and in [9], and their interesting properties has led to a large body of literature, see [6] and [13], and references therein. In applications, threshold graphs appear as models of social networks [19], in the problem of synchronizing parallel

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