Linear Algebra and its Applications 471 (2015) 615–635

Contents lists available at ScienceDirect

Linear Algebra and its Applications

www.elsevier.com/locate/laa

Trees with 4 or 5 distinct normalized Laplacian eigenvalues

Rodrigo O. Braga [a,*], Renata R. Del-Vecchio [b], Virgínia M. Rodrigues [c], Vilmar Trevisan [c]

[a] *UNISINOS, Universidade do Vale do Rio dos Sinos, Ciências Exatas e Tecnológicas, São Leopoldo, Brazil*
[b] *UFF, Universidade Federal Fluminense, Instituto de Matemática, Niterói, Brazil*
[c] *UFRGS, Universidade Federal do Rio Grande do Sul, Instituto de Matemática, Porto Alegre, Brazil*

A R T I C L E I N F O

Article history:
Received 21 December 2013
Accepted 17 January 2015
Available online 3 February 2015
Submitted by R. Brualdi

MSC:
05C50
15A18
05C85

Keywords:
Normalized Laplacian eigenvalues
Eigenvalue multiplicity
Small diameter trees

A B S T R A C T

We develop a tool for locating eigenvalues of the normalized Laplacian matrix of trees. This is obtained by extending an algorithm designed for the adjacency matrix, due to Jacobs and Trevisan (2011). As an application, we study the multiplicity of normalized Laplacian eigenvalues of small diameter trees. Our main result is the characterization of the trees that have 4 or 5 distinct normalized Laplacian eigenvalues. We also show that with a fixed diameter these trees are determined by their normalized Laplacian spectrum.

* Corresponding author.
E-mail addresses: rodrigob@unisinos.br (R.O. Braga), renata@vm.uff.br (R.R. Del-Vecchio), vrodrig@mat.ufrgs.br (V.M. Rodrigues), trevisan@mat.ufrgs.br (V. Trevisan).

http://dx.doi.org/10.1016/j.laa.2015.01.018

1. Introduction

The normalized Laplacian matrix of a graph, introduced by Chung in [4], has been extensively studied over the last years. Given $G(V, E)$, a simple undirected graph of order n, the normalized Laplacian matrix of G is the $n \times n$ matrix with rows and columns indexed by the vertices of G, whose entries are

$$\mathcal{L}(u,v) = \begin{cases} 1, & \text{if } u = v \text{ and } d_u > 0, \\ -\frac{1}{\sqrt{d_u \cdot d_v}}, & \text{if } \{u,v\} \in E, \\ 0, & \text{otherwise}, \end{cases}$$

where d_v denotes the degree of a vertex $v \in V$.

In his Ph.D. dissertation, Cavers [2, Section 2.3] showed that when dealing with random walks the normalized Laplacian matrix is a more natural tool than the widely used adjacency and combinatorial Laplacian matrices. Moreover, there are some properties of the spectrum that make the normalized Laplacian easier to study than other common matrices. For instance, the normalized spectrum of any graph G is in $[0, 2]$, and the largest eigenvalue of $\mathcal{L}(G)$ is 2 if and only if a connected component of G is a nontrivial bipartite graph. In addition, the spectrum of any bipartite graph is symmetric about 1, including multiplicities [4]. In the special case where the graph is a tree, it follows that 0 and 2 are simple eigenvalues. For a more detailed perspective about spectral properties of the normalized Laplacian matrix, we refer to Butler [1] and Cavers [2].

We are interested in investigating the normalized Laplacian spectrum of graphs having few distinct eigenvalues. A motivation for studying these graphs is given by van Dam and Omidi in [5], where they claim that most of these graphs are not determined by the spectrum, which means it is hard to distinguish them by the spectrum.

In [2, Corollary 2.6.4], Cavers proved that a connected graph G on $n \geq 3$ vertices has exactly 2 distinct $\mathcal{L}$-eigenvalues if and only if G is the complete graph. He also characterized the graphs that have exactly 3 distinct $\mathcal{L}$-eigenvalues, in the case they contain at least one pendant vertex. He showed that these graphs must have diameter 2, thus the only tree with this property is the star. In [5], van Dam and Omidi gave a combinatorial characterization of graphs whose normalized Laplacian has three distinct eigenvalues, without restrictions on the vertices of the graph G. Therefore, trees with exactly 2 or 3 distinct $\mathcal{L}$-eigenvalues are known: the complete graph on two vertices K_2 and the star S_n, $n > 2$, respectively. In this paper we characterize the trees that have exactly 4 or 5 distinct normalized Laplacian eigenvalues, and show that there are no isomorphic trees in this set with the same diameter. This means that the trees with 4 or 5 $\mathcal{L}$-eigenvalues and fixed diameter are determined by their spectrum.

We first notice that Chung [4] showed that if D is the diameter of a weighted graph G, then the normalized Laplacian matrix of G has at least $D + 1$ distinct eigen-