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Duality of matrix pencils, Wong chains and linearizations

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We consider two theoretical tools that have been introduced decades ago but whose usage is not widespread in modern literature on matrix pencils. One is *dual pencils*, a pair of pencils with the same regular part and related singular structures. They were introduced by V. Kublanovskaya in the 1980s. The other is *Wong chains*, families of subspaces, associated with (possibly singular) matrix pencils, that generalize Jordan chains. They were introduced by K.T. Wong in the 1970s. Together, dual pencils and Wong chains form a powerful theoretical framework to treat elegantly singular pencils in applications, especially in the context of linearizations of matrix polynomials.

We first give a self-contained introduction to these two concepts, using modern language and extending them to a more general form; we describe the relation between them and show how they act on the Kronecker form of a pencil and on spectral and singular structures (eigenvalues, eigenvectors and minimal bases). Then we present several new applications of these results to more recent topics in matrix pencil theory, including: constraints on the minimal indices of singular Hamiltonian and symplectic pencils, new sufficient conditions under which pencils in $\mathbb{L}_1$, $\mathbb{L}_2$ linearization spaces are strong linearizations, a new perspective on Fiedler pencils, and a link

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between the Möller–Stetter theorem and some linearizations of matrix polynomials.

1. Introduction

Consider a pair of matrices of the same size $R_0, R_1 \in \mathbb{C}^{n\times p}$. In matrix theory, a degree-1 matrix polynomial $R_1x + R_0 \in \mathbb{C}[x]^{n\times p}$ is known as a *matrix pencil* [18,19].

Two pencils $L(x) := L_1x + L_0 \in \mathbb{C}[x]^{m\times n}$ and $R(x) := R_1x + R_0 \in \mathbb{C}[x]^{n\times p}$ are *dual* if the following two conditions hold:

D1 $L_1R_0 = L_0R_1$,

D2 $\operatorname{rank}\left[\begin{smallmatrix} L_1 & L_0 \end{smallmatrix}\right] + \operatorname{rank}\left[\begin{smallmatrix} R_1 \\ R_0 \end{smallmatrix}\right] = 2n$.

In this case, we say that $L(x)$ is a *left dual* of $R(x)$, and, conversely, $R(x)$ is a *right dual* of $L(x)$. Two dual pencils have the same eigenvalues and regular Kronecker structure, while their singular parts (if any) are related in a precise way.

Given a pencil $R(x)$ and four complex numbers α, β, γ, δ with $\alpha\delta \neq \beta\gamma$, the *Wong chain* attached to the eigenvalue $\lambda := \frac{\alpha}{\beta} \in \mathbb{C} \cup \{\infty\}$ is the family of nested subspaces $\{0\} = \mathcal{W}_0^{(\lambda)} \subseteq \mathcal{W}_1^{(\lambda)} \subseteq \mathcal{W}_2^{(\lambda)} \subseteq \cdots$ defined by the following property: for each $k \geq 0$: $\mathcal{W}_{k+1}^{(\lambda)}$ is the preimage under the map $\alpha R_1 + \beta R_0$ of the space $(\gamma R_1 + \delta R_0)\mathcal{W}_k^{(\lambda)}$. This family depends only on $R(x)$ and λ, as we prove in the following. Wong chains are essentially a generalization of Jordan chains and can also be defined for singular pencils.

Wong chains have been introduced in [40], and only recently reappeared in the study of matrix pencils [4–6]; the definition that we use here is a generalized version. Dual pencils appear in [26, Section 1.3], where they are given the name of *consistent pencils* and some of their theoretical properties are stated. Moreover, one can recognize the use of duality (of regular pencils only, which is a less interesting case) in the study of doubling and inverse-free methods [2,9,31], as well as in the work [3], which gives an elegant algebraic theory of operations on matrix pencils.

Yet, these tools seem to be underused with respect to their potential and we would like to bring them back to the attention of the matrix pencil community. We will argue that they are an elegant device for the theoretical study of matrix pencils, that allows us to obtain new results and revisit old ones, greatly simplifying the treatment of singular cases.

The structure of the paper is the following: in Section 2, we recall some basic definitions and classical results on matrix pencils and matrix polynomials. In Sections 3 and 4 we introduce dual pencils and Wong chains and describe their properties, especially in relation to the Kronecker canonical form, eigenvectors and minimal bases. We then show how they can be used for several tasks in different applications: