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Restricted likelihood representation for multivariate heterogeneous random effects models

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A B S T R A C T

In the random effects model of meta-analysis for heterogeneous multidimensional data a canonical representation of the restricted likelihood function is obtained. This representation is related to a linear data transform which is based on the algebraic characteristics of error covariance matrices which are supposed to commute. The relationship between the heterogeneity covariance matrix estimators and the mean effect estimators is explored. It is noted that the sample mean exhibits the Stein-type phenomenon being an inadmissible estimator of the effect size under the quadratic loss when the number of studies exceeds three.

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1. Introduction: meta-analysis model

One of the important applications of random effects models is meta-analysis where one has to combine information in multivariate measurements made in several studies

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which commonly exhibit not only non-negligible between-study variability, but also have different within-study precision.

Consider a model where several independent sources provide the estimates of q-dimensional parameter θ (representing the treatment effect or the common mean). Let the i-th study vector estimate of θ be X_i, $i = 1, \ldots, n$. In the random effects model of meta-analysis

$$X_i = \theta + \ell_i + \epsilon_i, \tag{1}$$

where the independent vectors ℓ_i represent random *between-study* effects with zero mean and some unknown $q \times q$ covariance matrix Ξ (which may have rank smaller than q). If the errors ϵ_i are assumed to be independent and normally distributed, $\epsilon_i \sim N(0, S_i)$, then $X_i \sim N(\theta, \Xi + S_i)$.

This model appears under scenario where each study measures its linear functions of θ, i.e., when the i-th study data vector consists of n_i measurements,

$$Y_i = B_i[\theta + \ell_i] + \epsilon_i. \tag{2}$$

Here B_i is the known i-th laboratory design matrix having the rank q and the size $n_i \times q$. The meaning of θ, ℓ_i and ϵ_i remains the same as in (1), and statistics $X_i = (B_i^T B_i)^{-1} B_i^T Y_i$ (the classical least squares estimators) satisfy this model. Unlike the general mixed effects model, the condition in (2) is that the random between-study effect with probability one belongs to the space spanned by columns of B_i. See [9] for further motivation of (2) and for some examples.

In many applications, e.g. [4,5,8], estimates of the full covariance matrices S_i are not available but estimators V_i of the variances are given. In view of the lack of appropriate data, simplifying assumptions are to be made. For example, one may impose the condition that $S_i = V_i^{1/2} R V_i^{1/2}$ for some given correlation matrix R and a diagonal matrix V_i. Then the results obtained for several correlation matrices R can be compared (see Section 5). The assumption made in this work is that all given matrices S_i as well as unknown Ξ commute.

In the setting with known covariance matrices S_i's, the parameters to be estimated are the matrix Ξ and θ itself. If Ξ is known, the best linear unbiased estimator of θ is

$$\tilde{X} = \left[\sum_i (S_i + \Xi)^{-1}\right]^{-1} \sum_i (S_i + \Xi)^{-1} X_i, \tag{3}$$

so the traditional methods seek to estimate Ξ using a plug-in estimator of θ afterwards.

We discuss some of these traditional estimators in Section 3 where a wider class of θ-estimators is suggested. This class is motivated by the form of Bayes procedures and by the representation of the restricted likelihood function derived in Section 2. This canonical representation makes use of the polynomials determined by the matrices S_i, $i = 1, \ldots, n$.