Linear Algebra and its Applications 473 (2015) 202–216

Contents lists available at ScienceDirect

Linear Algebra and its Applications

www.elsevier.com/locate/laa

Singular value decomposition of the third multivariate moment

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ARTICLE INFO

Article history:
Received 11 July 2013
Accepted 28 May 2014
Available online 9 July 2014
Submitted by W.B. Wu

MSC:
15A18
60G09
60G10
60G51

Keywords:
Exchangeable process
Multivariate skewness
Linear process
Reversible process
Singular value decomposition

ABSTRACT

The third moment of a random vector is a matrix which conveniently arranges all moments of order three which can be obtained from the random vector itself. We investigate some properties of its singular value decomposition. In particular, we show that left eigenvectors corresponding to positive singular values of the third moment are vectorized, symmetric matrices. We derive further properties under the additional assumptions of exchangeability, reversibility and independence. Statistical applications deal with measures of multivariate skewness.

1. Introduction

Let $x = (X_1, \ldots, X_d)^T$ be a d-dimensional random vector satisfying $E(|X_i^3|) < +\infty$, for $i = 1, \ldots, d$. The third moment of x is the $d^2 \times d$ matrix $M_3(x) = E(x \otimes x^T \otimes x)$, where "$\otimes$" denotes the Kronecker product (see, for example, De Luca and Loperfido [5]).

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http://dx.doi.org/10.1016/j.laa.2014.05.043

In the following, when referring to the third moment of a random vector, we shall implicitly assume that all appropriate moments exist. The third central moment of x, also known as its third cumulant, is the third moment of $x - \mu$, where μ is the mean of x. Statistical applications of the third moment include, but are not limited to: factor analysis (Mooijaart [21]), density approximations (Van Hulle [24]), Independent Component Analysis (De Lathauwer et al. [4]), financial econometrics (De Luca and Loperfido [5]), cluster analysis (Loperfido [13]).

No one of the above authors studied the singular value decomposition (SVD, henceforth) of the third moment, which is a fundamental tool in both mathematics and statistics. The role of the SVD in mathematics is well reviewed by Martin and Porter [20]. In Statistics, the SVD provides the theoretical foundations for the biplot (Gower [8]), canonical correlation analysis (Mardia et al. [19, p. 283]) and correspondence analysis (Greenacre and Hastie [9]).

This paper examines the SVD of the third multivariate moment both in the general case and under additional assumptions. In the general case, it shows that left eigenvectors corresponding to positive singular values of the third moment are vectorized, symmetric matrices. Additional properties are derived for finite realizations of well-known stochastic processes, which nicely mirror those of their second moments. Finally, the paper shows that the SVD is instrumental in obtaining properties of main measures of multivariate skewness.

The rest of the paper is organized as follows. Section 2 discusses the SVD for the third moment, in the general case. Section 3 obtains some inequalities for measures of multivariate skewness. Sections 4 and 5 deal with third moments under independence and invariance assumptions, respectively. Section 6 shows that theorems and proofs similar to those in Section 2 also hold for fourth moments and cumulants. Section 7 applies results in Section 3 to financial data.

2. Decomposition

This section investigates the SVD and a related decomposition of the third multivariate moment. The symbols I_d, 1_d and 0_d will denote the $d \times d$ identity matrix, the d-dimensional vector of ones and the d-dimensional vector of zeros, respectively. Also, $vec\, A = vec(A)$ will denote the vectorization operator, which stacks the columns of the matrix A on top of one another (Rao and Rao [23, p. 200]). The row vector $vec^T A$ will denote the transpose of the vectorized matrix A, while the column vector $vec\, A^T$ will denote the vectorized transpose of A.

We shall now recall some fundamental properties of the Kronecker product which we shall use repeatedly in the following proofs (see, for example, Rao and Rao [23, pp. 194–201]): (P1) the Kronecker product is associative: $(A \otimes B) \otimes C = A \otimes (B \otimes C) = A \otimes B \otimes C$; (P2) if matrices A, B, C and D are of appropriate size, then $(A \otimes B)(C \otimes D) = AC \otimes BD$ and $(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$ if both A and B are invertible matrices; (P3) if a and b are two vectors, then ab^T, $a \otimes b^T$ and $b^T \otimes a$ denote the same matrix; (P4) for any