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Spectral filtering for trend estimation ☆

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ABSTRACT

This paper deals with trend estimation at the boundaries of a time series by means of smoothing methods. After deriving the asymptotic properties of sequences of matrices associated with linear smoothers, two classes of asymmetric filters that approximate a given symmetric estimator are introduced: the reflective filters and antireflective filters. The associated smoothing matrices, though non-symmetric, have analytically known spectral decomposition. The paper analyses the properties of the new filters and considers reflective and antireflective algebras for approximating the eigensystems of time series smoothing matrices. Based on this, a thresholding strategy for a spectral filter design is discussed.

1. Introduction

Let us consider the time series model,

$$y_t = \mu_t + \epsilon_t, \quad t = 1, \ldots, n,$$

where y_t is the observed time series, μ_t is the trend component, also termed the signal, and ϵ_t is the noise, or irregular, component. The signal μ_t can be a random or deter-

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ministic function of time whereas the most common assumption for the noise ϵ_t is that it follows a zero mean stationary process either white noise or/and Gaussian. The interest is in estimating μ_t based on the available information, with the aim of separating permanent movements from transitory oscillations. To this purpose, smoothing methods such as local polynomial regression or kernel regression are often applied (see [12, 14,18,20] among the others; see also [16], for a general treatment of signal extraction). These methods provide linear estimators of the trend based on weighted average of the observations, $\hat{\mu}_t = \sum_{j=-h}^{h} w_j y_{t+j}$ for $t = h+1, \ldots, n-h$, where $\{w_{-h}, \ldots, w_0, \ldots, w_h\}$ is a symmetric filter, $w_j = w_{-j}$, satisfying the unbiasedness condition with respect to a constant trend, $\sum_{j=-h}^{h} w_j = 1$.

Estimates for the trend at the first and last h time points are obtained with asymmetric and time varying filters. Notice that estimates for the trend in correspondence to the last h time points are crucial in current analysis. Moreover, statistical agencies often face the problem of approximating a given symmetric filter with a set of asymmetric ones [5]. A typical example is provided by the Census X11 and X11/X12 ARIMA software (see [6] and [13]), where the central trend is estimated with the symmetric Henderson filters [11], while surrogate filters for the trend at the boundaries are used (see [8] and [9]).

In practice, a filtering matrix S is applied to the time series of observations collected in an n-dimensional vector $\mathbf{y}$ to obtain the vector of estimates $\hat{\boldsymbol{\mu}} = S\mathbf{y}$. Specifically, $S \in \mathbb{R}^n$ is a matrix filter that assumes the following form

$$S = \left[\begin{array}{cc} S_P & O_{h\times(n-2h)} \\ \hline \multicolumn{2}{c}{S_I} \\ \hline O_{h\times(n-2h)} & S_F \end{array}\right] \tag{1}$$

where $S_I \in \mathbb{R}^{(n-2h)\times n}$ is the Toeplitz matrix associated with the symmetric filter:

$$S_I = \begin{bmatrix} w_{-h} & \cdots & w_0 & \cdots & w_h & & \\ & \ddots & \ddots & \ddots & \ddots & \ddots & \\ & & w_{-h} & \cdots & w_0 & \cdots & w_h \end{bmatrix},$$

where $w_{-j} = w_j$, that is the starting filters are symmetric, while $S_P \in \mathbb{R}^{h\times 2h}$ and $S_F \in \mathbb{R}^{h\times 2h}$ are the matrices associated with the asymmetric filters. The use of a symmetric filter implies that past and future observations have the same relevance for trend estimation. The same assumption requires that $S_P = J_h S_F J_{2h}$, where J_k denote the flip matrix such that $(J_k)_{ij} = 1$ if $i + j = k + 1$ and zero otherwise, for $i, j = 1, \ldots, k$.

Several methods have been proposed to treat the problem of current analysis based on asymmetric filters, that is to derive or specify S_F, see the recent discussion in [17]. When the spectral properties of the matrices associated with the linear trend estimators are of interest, two classes of filters deserve attention. These are the classes of reflective filters and antireflective filters. The relevant property that the matrices associated with reflective or antireflective filters shares is that they have analytically known eigenval-