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Integral circulant Ramanujan graphs via multiplicativity and ultrafriable integers

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ABSTRACT

Any *integral circulant graph* $\mathrm{ICG}(n, \mathcal{D})$ is characterised by its order n and a set $\mathcal{D}$ of positive divisors of n in such a way that it has vertex set $\mathbb{Z}/n\mathbb{Z}$ and edge set $\{(a, b) : a, b \in \mathbb{Z}/n\mathbb{Z}, \gcd(a - b, n) \in \mathcal{D}\}$. Such graphs are regular, and a connected ρ-regular graph G is called *Ramanujan* if the second largest modulus of the eigenvalues of the adjacency matrix of G is at most $2\sqrt{\rho - 1}$.

In 2010 Droll described all Ramanujan unitary Cayley graphs, i.e. graphs of type $X_n := \mathrm{ICG}(n, \{1\})$ having the Ramanujan property. Recently, Le and the author classified all Ramanujan graphs $\mathrm{ICG}(p^s, \mathcal{D})$ for prime powers p^s and arbitrary divisor sets $\mathcal{D}$. We greatly extend the established results to graphs $\mathrm{ICG}(n, \mathcal{D})$ with arbitrary n and multiplicative divisor set $\mathcal{D}$: (i) We derive a criterion (in terms of Euler's totient function) for $\mathrm{ICG}(n, \mathcal{D})$ to be Ramanujan. (ii) We prove that for all even integers $n > 2$ and a positive proportion of the odd integers n, namely those having a "dominating" prime power factor, there exists a multiplicative divisor set $\mathcal{D}$ such that $\mathrm{ICG}(n, \mathcal{D})$ is Ramanujan. (iii) We show that the set of odd n for which no Ramanujan graph $\mathrm{ICG}(n, \mathcal{D})$ with multiplicative divisor set $\mathcal{D}$ exists, viz. ultrafriable integers, has positive density as well. The proofs of (ii) and (iii) use methods from analytic number theory.

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1. Introduction and basic terminology

Recently Le and the author [13] classified all integral circulant Ramanujan graphs having prime power order, thus greatly extending a corresponding result by Droll [6] for the special subclass of unitary Cayley graphs. Ramanujan graphs, which were introduced by Lubotzky, Phillips and Sarnak [14] in 1988, have strong connectivity properties with a lot of applicatory consequences, e.g. the resolution of an extremal problem in communication network theory (cf. [3]), but they are also of importance in theoretical computer science (cf. [9,16]). Moreover, they are closely related to the Riemann hypothesis for the Ihara zeta-function concerning the distribution of primes in graphs (see [24]).

A connected ρ-regular graph G has the largest eigenvalue ρ and is called *Ramanujan* if G has at least three vertices and the second largest modulus of its eigenvalues

$$\Lambda(G) := \max\{|\lambda| : \ \lambda \in \mathrm{Spec}(G), \ |\lambda| \neq \rho\} \tag{1}$$

satisfies $\Lambda(G) \leq 2\sqrt{\rho - 1}$, where the eigenvalues of a graph G simply are the eigenvalues of its adjacency matrix, and the *spectrum* $\mathrm{Spec}(G)$ is the set of all these eigenvalues. Observe that for each ρ-regular connected graph G the largest eigenvalue ρ occurs with multiplicity 1, and consequently G has eigenvalues λ with $|\lambda| < \rho$ if G has at least 3 vertices (cf. [4]), which means that $\Lambda(G)$ is well defined under this mild restriction.

For *integral circulant graphs*, i.e. graphs having a circulant adjacency matrix with integral eigenvalues, lots of interesting results have been obtained in recent years (see [18] for references in general, and [13] for references concerning spectral properties in particular). Integral circulant graphs are generalisations of unitary Cayley graphs. By the works of Klotz and T. Sander [10] and So [20], they can be defined as follows: For a given integer $n > 1$ and a set $\mathcal{D} \subseteq D(n) := \{d > 0 : \ d \mid n\}$ of positive divisors of n the corresponding integral circulant graph $\mathrm{ICG}(n, \mathcal{D})$ has vertex set $\mathbb{Z}/n\mathbb{Z}$ and edge set $\{(a, b) : a, b \in \mathbb{Z}/n\mathbb{Z}, \gcd(a - b, n) \in \mathcal{D}\}$, where $\mathbb{Z}/n\mathbb{Z}$ denotes the additive group of residue classes mod n. Due to this characterisation integral circulant graphs are apparently regular, and it is not surprising that they show lots of arithmetical features. Since $\mathrm{ICG}(n, \mathcal{D})$ has loops in case $n \in \mathcal{D}$, it is usually assumed that $\mathcal{D} \subseteq D^*(n) := D(n) \setminus \{n\}$.

Unitary Cayley graphs are the integral circulant graphs of type $X_n := \mathrm{ICG}(n, \{1\})$. In 2010 Droll [6] classified all X_n which have the Ramanujan property and asked for a classification of Ramanujan graphs in larger families of integral circulant graphs. Recently, Le and the author [13] complemented Droll's result by drawing up a complete list of graphs $\mathrm{ICG}(p^s, \mathcal{D})$ having the Ramanujan property for arbitrary prime powers p^s and $\mathcal{D} \subseteq D^*(p^s)$. As a simple consequence of this we obtained that there is an integral circulant Ramanujan graph $\mathrm{ICG}(p^s, \mathcal{D})$ for each prime power $p^s \geq 3$ (cf. Proposition 3.1 (i)). As we shall see, well-known properties of Ramanujan sums imply the generalisation of that corollary to integral circulant graphs of arbitrary order (Theorem 2.1), but the divisor sets we use are non-multiplicative.