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Directed strongly regular graphs with rank 5

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ABSTRACT

From the parameters (n, k, t, λ, μ) of a directed strongly regular graph (dsrg) A. Duval (1988) [4] showed how to compute the eigenvalues and multiplicities of the adjacency matrix, and thus the rank of the adjacency matrix. For every rational number q, where $\frac{1}{5} \leq q \leq \frac{7}{10}$, there is a feasible (i.e., satisfying Duval's conditions) parameter set for a dsrg with rank 5 and with $\frac{k}{n} = q$.

In this paper we show that there exist a dsrg with such a feasible parameter set only if $\frac{k}{n}$ is $\frac{1}{5}$, $\frac{1}{3}$, $\frac{2}{5}$, $\frac{1}{2}$, $\frac{3}{5}$, or $\frac{2}{3}$. Every dsrg with rank 5 therefore has parameters of a known graph. The proof is based on an enumeration of 5×5 matrices with entries in $\{0, 1\}$.

1. Introduction

A directed strongly regular graph with parameters (n, k, t, λ, μ) is a k-regular directed graph on n vertices such that every vertex is on t 2-cycles (which may be thought of as undirected edges), and the number of paths of length 2 from a vertex x to a vertex y is λ if there is an edge directed from x to y and it is μ otherwise. Thus the adjacency matrix A satisfies

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$$A^2 = tI + \lambda A + \mu(J - I - A) \quad \text{and} \quad AJ = JA = kJ.$$

It is usually assumed that $0 < t < k$. These graphs were introduced by A. Duval [4], who also showed that the spectrum of A may be computed from the parameters.

In some cases 0 is an eigenvalue of large multiplicity and then the rank of A is small. In [8], we proved that there exists a dsrg with parameters (n, k, t, λ, μ) and with adjacency matrix of rank 3 if and only if the parameters are either $(6m, 2m, m, 0, m)$ or $(8m, 4m, 3m, m, 3m)$, for some integer m, and there exists one with rank 4 if and only if (n, k, t, λ, μ) is either $(6m, 3m, 2m, m, 2m)$ or $(12m, 3m, m, 0, m)$, for an integer m. For rank 3, this was proved independently in [6].

The main theorem in this paper is a characterization of parameters with rank 5.

Theorem 1. *There exists a directed strongly regular graph with parameters* (n, k, t, λ, μ) *and with adjacency matrix of rank* 5 *if and only if the parameter set is one of the following:* $(20m, 4m, m, 0, m)$, $(36m, 12m, 5m, 2m, 5m)$, $(10m, 4m, 2m, m, 2m)$, $(16m, 8m, 5m, 3m, 5m)$, $(20m, 12m, 9m, 6m, 9m)$, *or* $(18m, 12m, 10m, 7m, 10m)$, *for some positive integer* m.

For $n \leq 110$, Theorem 1 excludes existence of directed strongly regular graphs with the following parameters: $(45, 12, 4, 1, 4)$, $(45, 24, 16, 10, 16)$, $(49, 28, 20, 13, 20)$, $(64, 16, 5, 1, 5)$, $(80, 24, 9, 3, 9)$, $(80, 56, 49, 35, 49)$, $(81, 36, 20, 11, 20)$, $(90, 24, 8, 2, 8)$, $(90, 48, 32, 20, 32)$, $(98, 28, 10, 3, 10)$ and $(98, 56, 40, 26, 40)$. The problem of existence in these cases was previously unsolved, see [2].

Note that in these results we assume that $t < k$. A dsrg with $t = k$, eigenvalue 0 and with rank r is an undirected complete r-partite graph, that exists for every $r \geq 2$. In the following table we list the possible values of $\frac{k}{n}$ for which there exists a dsrg with rank $r \leq 5$, including $\frac{r-1}{r}$ that we get when $t = k$. We see that then the list is symmetric around $\frac{1}{2}$ for each $r \leq 5$. It would be interesting to know if this is also true for $r \geq 6$.

Rank	Values of k/n
2	$\frac{1}{2}$
3	$\frac{1}{3}, \frac{1}{2}, \frac{2}{3}$
4	$\frac{1}{4}, \frac{1}{2}, \frac{3}{4}$
5	$\frac{1}{5}, \frac{1}{3}, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, \frac{2}{3}, \frac{4}{5}$

For the proof of the *if* part of Theorem 1 we refer to known constructions. Duval [4] proved that if for one of the six families of parameters sets there exists a dsrg for $m = 1$ then there exists a dsrg for every parameter set in that family. This construction replaces each vertex by a set of m independent vertices, and it works when $t = \mu$. Graphs with parameters $(20, 4, 1, 0, 1)$ and $(10, 4, 2, 1, 2)$ were also constructed by Duval. A dsrg with parameters $(20, 7, 4, 3, 2)$ was constructed in [10]. Its complement has parameters $(20, 12, 9, 6, 9)$. A dsrg with parameters $(18, 5, 3, 2, 1)$ and the complementary