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Large vector spaces of block-symmetric strong linearizations of matrix polynomials

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A B S T R A C T

Given a matrix polynomial $P(\lambda) = \sum_{i=0}^{k} \lambda^i A_i$ of degree k, where A_i are $n \times n$ matrices with entries in a field $\mathbb{F}$, the development of linearizations of $P(\lambda)$ that preserve whatever structure $P(\lambda)$ might posses has been a very active area of research in the last decade. Most of the structure-preserving linearizations of $P(\lambda)$ discovered so far are based on certain modifications of block-symmetric linearizations. The block-symmetric linearizations of $P(\lambda)$ available in the literature fall essentially into two classes: linearizations based on the so-called Fiedler pencils with repetition, which form a finite family, and a vector space of dimension k of block-symmetric pencils, called $\mathbb{DL}(P)$, such that most of its pencils are linearizations. One drawback of the pencils in $\mathbb{DL}(P)$ is that none of them is a linearization when $P(\lambda)$ is singular. In this paper we introduce new vector spaces of block-symmetric pencils, most of which are strong linearizations of $P(\lambda)$. The dimensions of these spaces are $O(n^2)$, which, for $n \geq \sqrt{k}$, are much larger than the dimension of $\mathbb{DL}(P)$. When k is

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Vector space $\mathbb{DL}(P)$

odd, many of these vector spaces contain linearizations also when $P(\lambda)$ is singular. The coefficients of the block-symmetric pencils in these new spaces can be easily constructed as $k \times k$ block-matrices whose $n \times n$ blocks are of the form 0, $\pm\alpha I_n$, $\pm\alpha A_i$, or arbitrary $n \times n$ matrices, where α is an arbitrary nonzero scalar.

1. Introduction

Let $\mathbb{F}$ be an arbitrary field and $M_n(\mathbb{F})$ be the set of $n \times n$ matrices with entries in $\mathbb{F}$. Throughout this paper we consider the matrix polynomial of degree k and size $n \times n$

$$P(\lambda) = \sum_{i=0}^{k} \lambda^i A_i, \quad \text{where } A_0, \dots, A_k \in M_n(\mathbb{F}),\ A_k \neq 0. \tag{1.1}$$

The matrix polynomial $P(\lambda)$ is said to be singular if $\det P(\lambda)$ is identically zero, and it is said to be regular otherwise. Matrix polynomials arise in many applications, are receiving considerable attention in the literature in the last years, and some general references on this topic are [13,19,29].

The most extended way to deal in theory and applications with matrix polynomials is via *linearizations* [13]. A linearization of the matrix polynomial $P(\lambda)$ is a pencil $L(\lambda) = \lambda L_1 - L_0$ of size $(nk)\times(nk)$ such that there exist two unimodular matrix polynomials, i.e., matrix polynomials with constant nonzero determinant, $U(\lambda)$ and $V(\lambda)$, which satisfy

$$U(\lambda)\, L(\lambda)\, V(\lambda) = \begin{bmatrix} P(\lambda) & 0 \\ 0 & I_{n(k-1)} \end{bmatrix},$$

where $I_{n(k-1)}$ is the identity matrix of size $n(k-1) \times n(k-1)$. Even more interesting are the *strong linearizations* of $P(\lambda)$ [12], whose definition requires to introduce first the *reversal* of $P(\lambda)$ as the polynomial $\operatorname{rev} P(\lambda) := \lambda^k P(1/\lambda)$. Then a linearization $L(\lambda)$ of $P(\lambda)$ is said to be *strong* if $\operatorname{rev} L(\lambda)$ is also a linearization of $\operatorname{rev} P(\lambda)$. The key property of strong linearizations is that they have the same finite and infinite regular spectral structures as $P(\lambda)$ [11]. We emphasize that we are using the classical definitions and sizes of linearizations and strong linearizations. Extensions of these concepts allowing other sizes have been considered recently by several authors (see [11] and the references therein). In this paper, given a pencil $L(\lambda) = \lambda L_1 - L_0$, we will call the constant matrices L_0 and L_1 the coefficients of $L(\lambda)$ of degree 0 and 1, respectively.

The matrix polynomials that arise in applications have very often particular structures. For example, they can be symmetric, Hermitian, palindromic, or alternating, among many other possible structures (see [20,21,23,27] and the references therein for more details). Most structured matrix polynomials have spectra with particular symmetries or constrains. Therefore, it is important to have strong linearizations of such