Linear Algebra and its Applications 470 (2015) 40–50

Contents lists available at ScienceDirect

Linear Algebra and its Applications

www.elsevier.com/locate/laa

Singular points of the algebraic curves of symmetric hyperbolic forms

Mao-Ting Chien [a,*], Hiroshi Nakazato [b]

[a] Department of Mathematics, Soochow University, Taipei 11102, Taiwan
[b] Department of Mathematical Sciences, Faculty of Science and Technology, Hirosaki University, Hirosaki 036-8561, Japan

ARTICLE INFO

Article history:
Received 18 October 2013
Accepted 4 February 2014
Available online 22 February 2014
Submitted by I. Spitkovsky

Dedicated to Professor Leiba Rodman on the occasion of his 65th birthday

MSC:
14J17
11E20
15A60

Keywords:
Singular point
Hyperbolic form
Numerical range

ABSTRACT

We investigate singular points of the algebraic curve defined by a hyperbolic form satisfying some symmetric properties. We show that all singular points of such a curve are real nodes.

1. Introduction

Let A be an $n \times n$ matrix. The numerical range of A is defined as the set

$$W(A) = \{\xi^* A\xi\colon\ \xi \in \mathbb{C}^n,\ \xi^*\xi = 1\}.$$

* Corresponding author.
E-mail addresses: mtchien@scu.edu.tw (M.-T. Chien), nakahr@cc.hirosaki-u.ac.jp (H. Nakazato).

http://dx.doi.org/10.1016/j.laa.2014.02.006
0024-3795/

It is a classical result known as the Toeplitz–Hausdorff theorem that the numerical range $W(A)$ is a convex set. Kippenhahn [18] characterized the numerical range $W(A)$ as the convex hull of the real affine part of the dual curve of $F_A(t,x,y) = 0$, where $F_A(t,x,y)$ is the real ternary form associated with A defined by

$$F_A(t,x,y) = \det\bigl(tI_n + x\Re(A) + y\Im(A)\bigr),$$

where $\Re(A) = (A + A^*)/2$ and $\Im(A) = (A - A^*)/(2i)$. From the viewpoint of Kippenhahn ternary forms, the numerical range has been studied by many authors in diverse directions (cf. [2,4,11,12,15,17]).

A real ternary homogeneous form $F(t,x,y)$ of degree n is said to be hyperbolic with respect to $(1,0,0)$ if $F(1,0,0) \neq 0$ and the equation $F(t,\cos\theta,\sin\theta) = 0$ in t has n real solutions counting the multiplicities for any $0 \leqslant \theta \leqslant 2\pi$. This notion was originally introduced as the principal symbol of hyperbolic differential operators (cf. [13]). Determinantal representation of a real ternary hyperbolic form was conjectured by Fiedler [11] and Lax [19] in slightly different forms. Fiedler conjectured that a hyperbolic form $F(t,x,y)$ is realized as $F_A(t,x,y)$ for some complex matrix A, while Lax conjectured the matrix A is a complex symmetric matrix. Fiedler [12] himself gave a partial result in the case the curve $F(t,x,y) = 0$ is a rational curve. Helton and Vinnikov [16] proved the conjecture in Lax formulation (cf. also [20]). Constructions of the symmetric matrices of certain hyperbolic forms were given in [4,6]. The method to construct the symmetric matrices is also used in semi-definite optimization (cf. [22]).

For arbitrary complex numbers $a_1, a_2, \ldots, a_n$, we consider an $n \times n$ cyclic weighted shift matrix $S(a_1, a_2, \ldots, a_n)$ defined as

$$S = S(a_1, a_2, \ldots, a_n) = \begin{pmatrix} 0 & a_1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & a_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \vdots & \vdots & \ddots & a_{n-1} \\ a_n & 0 & \cdots & \cdots & \cdots & 0 \end{pmatrix}.$$

The numerical range of a cyclic weighted shift matrix has been studied by many authors, see, for instance, [7,9,14,24,25]. In particular, it is shown, in [7], that the algebraic curve C_F defined by a real ternary form $F_S(t,x,y)$ associated with a cyclic nonzero weighted shift matrix $S = S(a_1, a_2, \ldots, a_n)$ satisfies the following conditions:

(i) The form $F_S(t,x,y)$ is hyperbolic with respect to $(1,0,0)$ and $F(1,0,0) = 1$;
(ii) The form $F_S(t,x,y)$ is weakly circular symmetric, that is,

$$F_S\bigl(t, \cos(2\pi/n)x - \sin(2\pi/n)y, \sin(2\pi/n)x + \cos(2\pi/n)y\bigr) = F_S(t,x,y)$$

for any $[(t,x,y)] \in \mathbf{CP}^2$;