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Characteristic invariant subspaces generated by a single vector ☆

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ABSTRACT

If f is an endomorphism of a finite dimensional vector space V over a field K then an invariant subspace $X \subseteq V$ is called hyperinvariant (respectively, characteristic) if X is invariant under all endomorphisms (respectively, automorphisms) that commute with f. The characteristic hull of a subset W of V is defined to be the smallest characteristic subspace in V that contains W. It is known that characteristic subspaces that are not hyperinvariant can only exist when $|K| = 2$. In this paper we study subspaces X which are the characteristic hull of a single element. In the case where $|K| = 2$ we derive a necessary and sufficient condition such that X is hyperinvariant.

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1. Introduction

Let V be an n-dimensional vector space over a field K and let f be a K-endomorphism of V. In this paper we consider two types of invariant subspaces. A subspace $X \subseteq V$ is said to be *hyperinvariant* (under f) if it remains invariant under all endomorphisms of V that commute with f (see e.g. [12, p. 305]). If X is an f-invariant subspace of V and if X is invariant under all automorphisms of V that commute with f, then [2] we say that X is *characteristic* (with respect to f). Let $\mathrm{Inv}(V, f)$, $\mathrm{Hinv}(V, f)$, and $\mathrm{Chinv}(V, f)$ be sets of invariant, hyperinvariant and characteristic subspaces of V, respectively. These sets are lattices (with respect to set inclusion), and

$$\mathrm{Hinv}(V, f) \subseteq \mathrm{Chinv}(V, f) \subseteq \mathrm{Inv}(V, f).$$

Thus, if W is a subset of V then there is a smallest characteristic subspace that contains W. We denote it by $\langle W\rangle^c$ and call it the *characteristic hull* of W. The structure of the lattice $\mathrm{Hinv}(V, f)$ is well understood ([14,9,15], [12, p. 306]). We point out that $\mathrm{Hinv}(V, f)$ is the sublattice of $\mathrm{Inv}(V, f)$ generated by

$$\mathrm{Ker}\, f^k,\ \mathrm{Im}\, f^k, \quad k = 0, 1, \ldots, n.$$

If the characteristic polynomial of f splits over K (such that all eigenvalues of f are in K) then one can restrict the study of hyperinvariant and of characteristic subspaces to the case where f has only one eigenvalue, and therefore to the case where f is nilpotent. Thus, throughout this paper we shall assume $f^n = 0$. Let $\Sigma(\lambda) = \mathrm{diag}(1, \ldots, 1, \lambda^{t_1}, \ldots, \lambda^{t_m}) \in K^{n\times n}[\lambda]$ be the Smith normal form of f such that $t_1 + \cdots + t_m = n$. We say that an elementary divisor λ^r is *unrepeated* if it appears exactly once in $\Sigma(\lambda)$. Otherwise λ^r is said to be *repeated*. Thus λ^r is unrepeated (see e.g. [13, p. 27]) if and only if $\dim(V[f] \cap f^{r-1}V/V[f] \cap f^rV) = 1$. We call a vector $u \in V$ a *generator* if the f-cyclic subspace generated by u has an f-invariant complement in V.

It is known ([16], [13, pp. 63–64], [2]) that all characteristic subspaces are hyperinvariant if the field K has more than two elements. Hence only if $K = GF(2)$ one may find K-endomorphisms f of V with a characteristic subspace that is not hyperinvariant. According to Shoda [16, p. 619] such subspaces exist if and only if f has two unrepeated elementary divisors λ^r and λ^s such that r and s are non-consecutive natural numbers (see also [5, Theorem 9, p. 510] and [13, pp. 63–64]). In this paper we study subspaces $X = \langle z\rangle^c$, which are the characteristic hull of a single element z, and in the case where $|K| = 2$ we derive a necessary and sufficient condition such that X is hyperinvariant.

We now describe the content of our paper. Having introduced the required definitions we state an essential part of our main result in Section 1.1. Basic facts on height and exponent are contained in Section 1.2. Our study relies on results of generators in Section 2.1, properties of hyperinvariant subspaces in Section 2.2 and on Baer's decomposition lemma in Section 2.3. Auxiliary results in Section 3 prepare the proof of