Linear Algebra and its Applications 470 (2015) 120–184

Contents lists available at ScienceDirect

Linear Algebra and its Applications

www.elsevier.com/locate/laa

Möbius transformations of matrix polynomials

D. Steven Mackey [a,1], Niloufer Mackey [a,1], Christian Mehl [b,*,2], Volker Mehrmann [b,2]

[a] Department of Mathematics, Western Michigan University, Kalamazoo, MI 49008, USA
[b] Institut für Mathematik, MA 4-5, Technische Universität Berlin, 10623 Berlin, Germany

ARTICLE INFO

Article history:
Received 9 January 2014
Accepted 8 May 2014
Available online 2 June 2014
Submitted by J. Behrndt

Dedicated to Leiba Rodman on the occasion of his 65th birthday

MSC:
65F15
15A18
15A21
15A54
15A57

Keywords:
Möbius transformation
Generalized Cayley transform
Matrix polynomial
Matrix pencil
Smith form

ABSTRACT

We discuss Möbius transformations for general matrix polynomials over arbitrary fields, analyzing their influence on regularity, rank, determinant, constructs such as compound matrices, and on structural features including sparsity and symmetry. Results on the preservation of spectral information contained in elementary divisors, partial multiplicity sequences, invariant pairs, and minimal indices are presented. The effect on canonical forms such as Smith forms and local Smith forms, on relationships of strict equivalence and spectral equivalence, and on the property of being a linearization or quadratification are investigated. We show that many important transformations are special instances of Möbius transformations, and analyze a Möbius connection between alternating and palindromic matrix polynomials. Finally, the use of Möbius transformations in solving polynomial inverse eigenproblems is illustrated.

* Corresponding author.
E-mail addresses: steve.mackey@wmich.edu (D.S. Mackey), nil.mackey@wmich.edu (N. Mackey), mehl@math.tu-berlin.de (C. Mehl), mehrmann@math.tu-berlin.de (V. Mehrmann).

1 Supported by National Science Foundation grants DMS-0713799 and DMS-1016224. Support from Deutsche Forschungsgemeinschaft through DFG Research Center Matheon 'Mathematics for key technologies' in Berlin, FZT 86-C29 during research visits to TU Berlin is gratefully acknowledged.

2 Supported by Deutsche Forschungsgemeinschaft through DFG Research Center Matheon 'Mathematics for key technologies' in Berlin, FZT 86-C29.

http://dx.doi.org/10.1016/j.laa.2014.05.013

Local Smith form
Elementary divisors
Partial multiplicity sequence
Jordan characteristic
Jordan chain
Invariant pair
Compound matrices
Minimal indices
Minimal bases
Structured linearization
Palindromic matrix polynomial
Alternating matrix polynomial

1. Introduction

The fundamental role of functions of the form $f(z) = (az+b)/(cz+d)$ in the theory of analytic functions of a complex variable is well-established and classical. Such functions are variously known as fractional linear rational functions [1,17,58], bilinear transformations [7,54–56] or more commonly, Möbius functions. A particularly important example is the Cayley transformation, see e.g., [54] or the variant in [28,50], which extends easily to matrix pencils or polynomials. The Cayley transformation is widely used in many areas, such as in the stability analysis of continuous and discrete-time linear systems [33, 36], in the analysis and numerical solution of discrete-time and continuous-time linear–quadratic optimal control problems [50,59], and in the analysis of geometric integration methods [34].

The main goal of this paper is to present a careful study of the influence of Möbius transformations on properties of general matrix polynomials over arbitrary fields. These include regularity, rank, determinant, constructs such as the compounds of matrix polynomials, and structural properties such as sparsity and symmetry. We show when spectral information contained in elementary divisors, partial multiplicity sequences, invariant pairs, minimal indices, and minimal bases is preserved, or how its change can be tracked. We study the effect on canonical forms such as Smith forms and local Smith forms, on the relations of strict equivalence and spectral equivalence, and on the property of being a linearization or quadratification. Many of the results presented here are fundamental in that they hold for all matrix polynomials, regular and singular, square and rectangular.

A variety of transformations exploited in the literature [2,16,18,29,35,44–47,51] will be seen to be special instances of Möbius transformations. The broader theory we present here generalizes and unifies results that were hitherto observed for particular transformations, and provides a more versatile tool for investigating fundamental aspects of matrix polynomials. Important applications include determining the relationships between various classes of structured matrix polynomials (alternating and palindromic, for example), investigating the definiteness of Hermitian matrix polynomials [2], numerical methods for the solution of structured eigenvalue problems and continuous-time Riccati equations via doubling algorithms, (see e.g., [31,32,52] and the references therein), the modeling of