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Some generalized numerical radius inequalities for Hilbert space operators

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We generalize several inequalities involving powers of the numerical radius for product of two operators acting on a Hilbert space. For any $A, B, X \in \mathbb{B}(\mathscr{H})$ such that A, B are positive, we establish some numerical radius inequalities for $A^\alpha X B^\alpha$ and $A^\alpha X B^{1-\alpha}$ $(0 \leq \alpha \leq 1)$ and Heinz means under mild conditions.

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1. Introduction

Let $(\mathscr{H}, \langle\cdot,\cdot\rangle)$ be a complex Hilbert space and $\mathbb{B}(\mathscr{H})$ denote the C^*-algebra of all bounded linear operators on $\mathscr{H}$. An operator $A \in \mathbb{B}(\mathscr{H})$ is called positive if $\langle Ax, x\rangle \geq 0$ for all $x \in \mathscr{H}$. We write $A \geq 0$ if A is positive. The numerical radius of $A \in \mathbb{B}(\mathscr{H})$ is defined by

$$w(A) = \sup\bigl\{\bigl|\langle Ax, x\rangle\bigr| : x \in \mathscr{H}, \|x\| = 1\bigr\}.$$

It is well known that $w(\cdot)$ defines a norm on $\mathbb{B}(\mathscr{H})$, which is equivalent to the usual operator norm $\|\cdot\|$. In fact, for any $A \in \mathbb{B}(\mathscr{H})$,

$$\frac{1}{2}\|A\| \leq w(A) \leq \|A\|. \tag{1.1}$$

Also if $A \in \mathbb{B}(\mathscr{H})$ is self-adjoint, then $w(A) = \|A\|$.

An important inequality for $w(A)$ is the power inequality stating that

$$w\bigl(A^n\bigr) \leq w^n(A)$$

for $n = 1, 2, \ldots$.

Several numerical radius inequalities improving the inequalities in (1.1) have been recently given in [2,3,8].

For instance, Dragomir proved that for any $A, B \in \mathbb{B}(\mathscr{H})$,

$$w^2(A) \leq \frac{1}{2}\bigl(w\bigl(A^2\bigr) + \|A\|^2\bigr), \tag{1.2}$$

and

$$w^r\bigl(B^*A\bigr) \leq \frac{1}{2}\bigl\|\bigl(A^*A\bigr)^r + \bigl(B^*B\bigr)^r\bigr\| \tag{1.3}$$

for all $r \geq 1$. The above inequalities can be found in [3,1], respectively. Some other interesting inequalities for numerical radius can be found in [8–10].

In Section 2 of this paper, we first generalize inequalities (1.2) and (1.3). Our generalization of inequality (1.3) in a particular case is sharper than this inequality.

In Section 3 we obtain numerical radius inequalities for Hilbert space operators $A^\alpha X B^\alpha$ and $A^\alpha X B^{1-\alpha}$ under conditions $A, B \geq 0$ and $0 \leq \alpha \leq 1$. We also find a numerical radius inequality for Heinz means.

2. Numerical radius inequalities for products of two operators

To prove our generalized numerical radius inequalities, we need several well-known lemmas. The first lemma is a simple consequence of the classical Jensen and Young inequalities (see [5]).