Linear Algebra and its Applications 472 (2015) 54–80

Contents lists available at ScienceDirect

Linear Algebra and its Applications

www.elsevier.com/locate/laa

Extensions of positive operators and functionals

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A R T I C L E   I N F O

Article history:
Received 20 September 2014
Accepted 22 January 2015
Available online 10 February 2015
Submitted by P. Semrl

MSC:
47B25
47B65

Keywords:
Positive operators
Extensions
Positive functionals
Representable functionals
Banach *-algebra

A B S T R A C T

We consider linear operators defined on a subspace of a complex Banach space into its topological antidual acting positively in a natural sense. The main theorem is a constructive characterization of the bounded positive extendibility of these linear mappings. In this frame we characterize operators with a compact or a closed range extension. As a main application of our general extension theorem, we present some necessary and sufficient conditions for a positive functional defined on a left ideal of a Banach *-algebra to admit a representable positive extension.

1. Introduction and preliminaries

Positive operators and their positive extensions play a significant role in the theory of Hilbert spaces and other areas of mathematics. Since there is a canonical conjugate isometric isomorphism between a Hilbert space and its topological dual via the Riesz representation theorem, we may investigate operators between a Banach space and its

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http://dx.doi.org/10.1016/j.laa.2015.01.028

topological antidual (see below). It will turn out that the concept of positivity can be naturally defined for such operators, which includes the Hilbert space case, as well.

We will see in Example 1.5 that this kind of positive operators naturally appear in the theory of positive functionals on Banach *-algebras. Since these functionals are fundamental tools for the *-representations of Banach *-algebras, we will study these operators more closely later in the paper.

Before we introduce our motivations and goals, we present our terminology for positive operators. Throughout this paper, let a complex Banach space E be given. We will denote by $\bar{E}'$ the *topological antidual* of E, that is, $\bar{E}'$ consists of all continuous mappings φ of E into the complex plane $\mathbb{C}$, which have the following properties:

$$\varphi(x+y) = \varphi(x) + \varphi(y), \quad x, y \in E,$$
$$\varphi(\lambda x) = \bar{\lambda}\varphi(x), \quad x \in E,\ \lambda \in \mathbb{C}.$$

The elements φ of $\bar{E}'$ are called *continuous anti-linear functionals* on E. For $x \in E$ and $\varphi \in \bar{E}'$ we set

$$\langle \varphi, x \rangle := \varphi(x).$$

It is clear that $\bar{E}'$ is a vector space and that

$$\|\varphi\| = \sup\bigl\{ \bigl|\langle \varphi, x \rangle\bigr| \bigm| x \in E,\ \|x\| \le 1 \bigr\}$$

defines a norm on $\bar{E}'$, such that $\bar{E}'$ is a Banach space with respect to this norm. Indeed, the following canonical mapping

$$E' \to \bar{E}', \quad f \mapsto \bar{f},$$

from the *topological dual* into the antidual of E is one-to-one, onto, anti-linear and isometric with respect to the corresponding norms. The topological antidual of $\bar{E}'$, called the *topological anti-bidual* of E will be denoted by $\bar{E}''$, so that E can be isometrically embedded into $\bar{E}''$ along the linear mapping $j_E : E \to \bar{E}'', x \mapsto \widehat{x}$ where

$$\langle \widehat{x}, \varphi \rangle := \widehat{x}(\varphi) = \overline{\langle \varphi, x \rangle}, \quad x \in E,\ \varphi \in \bar{E}'.$$

If another complex Banach space F is given, the (*anti-*)*adjoint* of a continuous linear operator $T \in \mathscr{B}(E; F)$ is determined along the corresponding antidualities

$$\bigl\langle T^* y', x \bigr\rangle = \bigl\langle y', Tx \bigr\rangle, \quad x \in E,\ y' \in \bar{F}', \tag{1.1}$$

so that T^* acts as a continuous linear operator between $\bar{F}'$ and $\bar{E}'$ with norm $\|T^*\| = \|T\|$.

Our main interest in this paper is linear operators A from a linear subspace $\operatorname{dom} A$ of E into $\bar{E}'$ satisfying