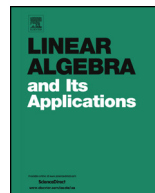




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The null space theorem



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ABSTRACT

The following results are proved.

Theorem 0.1 (*The Null Space Theorem*). Let X, Y be vector spaces, $P \in L(X), Q \in L(Y)$ be projections and $T \in L(X, Y)$ be invertible. (The restriction of QTP to $R(P)$ can be viewed as a linear operator from $R(P)$ to $R(Q)$. This is called a section of T by P and Q and will be denoted by $T_{P,Q}$.) Then there is a linear bijection between the null space of the section $T_{P,Q}$ of T and the null space of its complementary section T_{I_Y-Q, I_X-P} of T^{-1} .

Theorem 0.2. Let X be a Banach space with a Schauder basis $A = \{a_1, a_2, \dots\}$. Let T be a bounded (continuous) linear operator on X . Suppose the matrix of T with respect to A is tridiagonal. If T is invertible, then every submatrix of the matrix of T^{-1} with respect to A that lies on or above the main diagonal (or on or below the main diagonal) is of rank ≤ 1 .

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1. Introduction

This note has two objectives. The first is to make the Nullity Theorem known more widely. The second is to consider its generalization to infinite dimensional spaces and some applications of this generalization. We begin with some motivation for the Nullity Theorem.

Recall that a square matrix $A = [\alpha_{ij}]$ of order n is called *tridiagonal* if

$$\alpha_{ij} = 0 \quad \text{for } |i - j| > 1$$

Such a matrix is described completely by $3n - 2$ numbers (n on the main diagonal and $n - 1$ on each of superdiagonal and subdiagonal). In general, if a tridiagonal matrix is invertible, its inverse need not be tridiagonal. However, we may still expect that the inverse can be described completely by $3n - 2$ parameters. This is indeed true. It is known that if A is a tridiagonal matrix of order n and if A is also invertible, then every submatrix of A^{-1} that lies on or above the main diagonal is of rank ≤ 1 . Similar statement is true of submatrices lying on or below the main diagonal. This result is known at least since 1979 (see [2]). Several proofs of this result are available in the literature. The article [7] contains some of these proofs, references to these and other proofs and also a brief history and comments about possible generalizations.

In view of this result, the inverse can be described using $3n - 2$ parameters as follows: To start with we can choose $4n$ numbers $a_j, b_j, c_j, d_j, j = 1, \dots, n$ such that

$$\begin{aligned} (A^{-1})_{ij} &= a_i b_j \quad \text{for } i \leq j \quad \text{and} \\ &= c_i d_j \quad \text{for } j \leq i \end{aligned}$$

These $4n$ numbers have to satisfy the following constraints

$$a_i b_i = c_i d_i \quad \text{for } i = 1, \dots, n \text{ and } a_1 = 1 = b_1.$$

One proof of this theorem depends on the Nullity Theorem. This theorem uses the idea of complementary submatrices. Let A and B be square matrices of order n . Suppose M is a submatrix of A and N is a submatrix of B . We say that M and N are *complementary* if row numbers not used in one are the column numbers used in the other. More precisely, let I and J be subsets of the set $\{1, 2, \dots, n\}$ and let I^c denote the complement of I . Let $A(I, J)$ denote the submatrix of A obtained by choosing rows in I and columns in J . Then $A(I, J)$ and $B(J^c, I^c)$ are complementary submatrices. With this terminology, the Nullity Theorem has a very simple formulation.

Theorem 1.1 (Nullity Theorem). *Complementary submatrices of a square matrix and its inverse have the same nullity.*

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