Linear Algebra and its Applications 472 (2015) 135–141

Contents lists available at ScienceDirect

Linear Algebra and its Applications

www.elsevier.com/locate/laa

A Hadamard-type lower bound for symmetric diagonally dominant positive matrices

Christopher J. Hillar [a,*,1], Andre Wibisono [b,*]

[a] *Redwood Center for Theoretical Neuroscience, University of California, Berkeley, United States*
[b] *Department of Electrical Engineering and Computer Sciences, University of California, Berkeley, United States*

A R T I C L E   I N F O

Article history:
Received 8 April 2014
Accepted 8 January 2015
Available online 13 February 2015
Submitted by B. Lemmens

MSC:
15B48
15A15
15A45

Keywords:
Diagonally dominant
Positive matrices
Determinantal inequality
Hadamard's inequality

A B S T R A C T

We prove a new lower-bound form of Hadamard's inequality for the determinant of diagonally dominant positive matrices.

* Corresponding authors.
E-mail addresses: chillar@berkeley.edu (C.J. Hillar), wibisono@eecs.berkeley.edu (A. Wibisono).

[1] Partially supported by NSF grant IIS-0917342 and an NSF All-Institutes Postdoctoral Fellowship administered by the Mathematical Sciences Research Institute through its core grant DMS-0441170.

http://dx.doi.org/10.1016/j.laa.2015.01.037

1. Introduction

An $n \times n$ real matrix J is *diagonally dominant* if

$$\Delta_i(J) := |J_{ii}| - \sum_{j \neq i} |J_{ij}| \geq 0, \quad \text{for } i = 1, \ldots, n.$$

A particularly interesting case is when $\Delta_i(J) = 0$ for all i; we call such matrices *diagonally balanced*. Irreducible, diagonally dominant matrices are always invertible, and such matrices arise often in theory and applications. In this Note we study bounds on the determinant of symmetric diagonally dominant matrices that have positive entries. These matrices are always positive definite (e.g., by Lemma 2.1).

It is classical that the determinant of a positive semidefinite matrix A is bounded above by the product of its diagonal entries:

$$0 \leq \det(A) \leq \prod_{i=1}^{n} A_{ii}.$$

This well-known result is sometimes called Hadamard's inequality [5, Theorem 7.8.1]. A lower bound of this form, however, is not possible without additional assumptions. Surprisingly, there is such an inequality when J is diagonally dominant with positive entries.

Theorem 1.1. *Let $n \geq 3$, and let J be an $n \times n$ symmetric diagonally dominant matrix with off-diagonal entries $m \geq J_{ij} \geq \ell > 0$. Then, the following inequality holds:*

$$\frac{\det(J)}{\prod_{i=1}^{n} J_{ii}} \geq \left(1 - \frac{1}{2(n-2)}\sqrt{\frac{m}{\ell}}\left(1 + \frac{m}{\ell}\right)\right)^{n-1}$$
$$\to \exp\left(-\frac{1}{2}\sqrt{\frac{m}{\ell}}\left(1 + \frac{m}{\ell}\right)\right) \quad \text{as } n \to \infty.$$

The result above was discovered in an attempt to prove the following difficult norm inequality [4]. Let $S = (n-2)I_n + \mathbf{1}_n\mathbf{1}_n^\top$ be the diagonally balanced matrix whose off-diagonal entries are all equal to 1 (I_n is the $n \times n$ identity matrix and $\mathbf{1}_n$ is the n-dimensional column vector consisting of all ones).

Theorem 1.2. *(See [4].) Let $n \geq 3$. For any symmetric diagonally dominant matrix J with $J_{ij} \geq \ell > 0$, we have*

$$\|J^{-1}\|_\infty \leq \frac{1}{\ell}\|S^{-1}\|_\infty = \frac{3n-4}{2\ell(n-2)(n-1)}.$$

Moreover, equality is achieved if and only if $J = \ell S$.