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Extracting a basis with fixed block inside a matrix

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ABSTRACT

Given U an $n \times m$ matrix of rank n whose columns are denoted by $(u_j)_{j \leqslant m}$, several authors have already considered the problem of finding a subset $\sigma \subset \{1, \ldots, m\}$ such that $(u_i)_{i \in \sigma}$ span $\mathbb{R}^n$ and $\sqrt{\operatorname{Tr}((\sum_{i\in\sigma} u_i u_i^t)^{-1})}$ is minimized. In this paper, we generalize this problem by selecting arbitrary rank matrices instead of rank 1 matrices. Another generalization is considering the same problem while allowing a part of the matrix to be fixed. The methods of selection employed develop into algorithms.

1. Introduction

Let U be an $n \times m$ matrix; we see U as an operator from l_2^m to l_2^n. We denote by $\|U\|$ the operator norm of U, while the Hilbert–Schmidt norm of U is given by $\|U\|_{\mathrm{HS}} = \sqrt{\operatorname{Tr}(UU^t)}$. The stable rank of U is given by $\operatorname{srank}(U) := \|U\|_{\mathrm{HS}}^2 / \|U\|^2$. Note that the stable rank is always less or equal to the rank. We denote by $s_{\max}$ and $s_{\min}$ the largest and the smallest singular value, respectively. Given $\sigma \subset \{1, \ldots, m\}$, we denote by U_σ the restriction of U to the columns with indices in σ, i.e., $U_\sigma = UP_\sigma^t$ where

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$P_\sigma : \mathbb{R}^m \longrightarrow \mathbb{R}^\sigma$ is the canonical coordinate projection. When $\sigma = \emptyset$, we have $U_\sigma = 0$. Finally, if A and B are $n \times n$ symmetric matrices, the denotation $A \preceq B$ means that $B - A$ is positive semidefinite.

Column subset selection usualy refers to extracting from a matrix a column submatrix that has some distinguished properties. First results on column selection problems were obtained by Kashin [9]. The aim of the selection was to find a submatrix which minimizes the operator norm among all restrictions of the same size. It was later sharpened in [11] and [8] where, for any $n \times m$ matrix U and any $\lambda \leqslant 1/4$, it was proved that there exists $\sigma \subset \{1, \dots, m\}$ of size λm with $\|U_\sigma\| \leqslant C(\sqrt{\lambda}\|U\| + \|U\|_{\mathrm{HS}}/\sqrt{m})$, with C being a universal constant. In [4], Bourgain and Tzafriri considered selecting a block of columns which is well invertible, i.e., whose smallest singular value is bounded away from zero; their result states that there exist universal constants C and C' such that for any $n \times n$ matrix A whose columns are of norm 1, one can find $\sigma \subset \{1, \dots, m\}$ of size at least $C \operatorname{srank}(A)$ such that $s_{\min}(A_\sigma) \geqslant C'$. Later, Vershynin [15] extended the restricted invertibility principle of Bourgain and Tzafriri to the case of rectangular matrices. Moreover, he also studied the extraction of a well conditioned submatrix and proved that for any $\varepsilon \in (0, 1)$ and any $n \times m$ matrix U, there exists $\sigma \subset \{1, \dots, m\}$ of size at least $(1 - \varepsilon) \operatorname{srank}(U)$ such that $s_{\max}(U_\sigma)/s_{\min}(U_\sigma) \leqslant \varepsilon^{c \log(\varepsilon)}$. These results were very important in geometric functional analysis and had several applications. However, the proofs were not constructive as they were based on random selection and made use of Grothendieck's factorization theorem. In [14], Tropp was able to provide a randomized polynomial time algorithm to achieve the selection promised by the results of Bourgain and Tzafriri, and Kashin and Tzafriri. In [13], Spielman and Srivastava produced a deterministic polynomial time algorithm to find a well invertible submatrix inside any rectangular matrix, generalizing and improving the restricted invertibility principle of Bourgain and Tzafriri. Their proof is inspired by the method developed by Batson, Spielman and Srivastava [3] to find a spectral sparsifier of a graph. In that paper, they gave a deterministic polynomial time algorithm to find a submatrix, with much fewer columns, which approximates the singular values of the original rectangular matrix. More precisely, for any $\varepsilon \in (0, 1)$ and any $n \times m$ matrix U, they showed the existence of $\sigma \subset \{1, \dots, m\}$ of size $O(n/\varepsilon^2)$ such that $(1 - \varepsilon)UU^t \preceq U_\sigma U_\sigma^t \preceq (1 + \varepsilon)UU^t$. The main idea is to select the columns one by one, study the evolution of the singular values and keep controlling this evolution until the extraction is done. This idea was exploited in [17] to give a deterministic polynomial time algorithm which finds the submatrix promised by the result of Kashin and Tzafriri. Similar tools were developed in [16] in order to extract a well conditioned submatrix improving the result obtained by Vershynin [15]. More precisely, we proved that for any $\varepsilon \in (0, 1)$ and any $n \times m$ matrix U with columns of norm 1, there exists $\sigma \subset \{1, \dots, m\}$ of size at least $(1 - \varepsilon)^2 \operatorname{srank}(U)$ such that the singular values of U_σ lie between $\varepsilon/(2 - \varepsilon)$ and $(2 - \varepsilon)/\varepsilon$. When ε is close to 1, this result can be seen as dual to the one in [3] and allowed us to produce a deterministic polynomial time algorithm to partition any $n \times n$ zero diagonal matrix into $\log(n)$ square blocks of small norm around the diagonal. Doing such a partition with a number of blocks independent of the dimension is known as the