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Helly type theorems for the sum of vectors in a normed plane

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ABSTRACT

The main results here are two Helly type theorems for the sum of (at most) unit vectors in a normed plane. Also, we give a new characterization of centrally symmetric convex sets in the plane.

1. Main results

This paper is about the sum of vectors in a normed plane. We fix a norm $\|\,.\,\|$ in $\mathbb{R}^2$ whose unit ball is B; so B is a 0-symmetric convex body. There are some interesting

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results about sums of unit vectors in normed planes. For instance, it is proved by Swanepoel in [5] (and reproved later in [1]) that for every subset $V = \{v_1, \dots, v_n\} \subset B$ of unit vectors, with n an odd number, we may choose numbers $\epsilon_1, \epsilon_2, \dots, \epsilon_n$ from $\{1, -1\}$ such that $\|\sum_{v_i \in V} \epsilon_i v_i\| \le 1$. This time we are interested in unit vectors whose sum has length at least 1.

We write $u \cdot v$ for the usual scalar product of $u, v \in \mathbb{R}^2$ and $[n]$ for the set $\{1, 2, \dots, n\}$. Here comes our first result.

Theorem 1. *Assume $n \ge 3$ is an odd integer and $V = \{v_1, v_2, \dots, v_n\} \subset \mathbb{R}^2$ is a set of unit vectors. If $u \cdot v_i \ge 0$ for every $i \in [n]$ with a suitable non-zero vector $u \in \mathbb{R}^2$, then*

$$\|v_1 + v_2 + ... + v_n\| \ge 1.$$

Here and in what follows we can assume that V is a multiset, that is, $v_i = v_j$ can happen even if $i \ne j$. Perhaps one should think of V as a sequence of n vectors from $\mathbb{R}^2$.

In accordance to the celebrated Helly's theorem (see [3]), results of the type "if every m members of a family of objects have property P then the entire family has the property P" are called Helly-type theorems. Our main results are two unusual Helly type theorems whose proof uses Theorem 1. For information about Helly type results the reader may consult [4].

Theorem 2. *Assume $n \ge 3$ is an odd integer and $V = \{v_1, v_2, \dots, v_n\} \subset \mathbb{R}^2$ is a set of unit vectors. If the sum of any three of them has norm at least 1, then*

$$\|v_1 + v_2 + ... + v_n\| \ge 1.$$

Theorem 3. *Assume $n \ge 3$ is an odd integer and $V = \{v_1, v_2, \dots, v_n\} \subset B$. If the sum of any three elements of V has norm larger than 1, then*

$$\|v_1 + v_2 + ... + v_n\| > 1.$$

To our surprise Theorem 3 fails in the following form: If $V \subset B$, $|V|$ is odd, and the sum of any three of its elements has norm at least 1, then $\|v_1 + v_2 + ... + v_n\| \ge 1$. The example is with the max norm and the vectors are $v_1 = (1, 1)$, $v_2 = (-1, 1)$, and $v_3 = v_4 = v_5 = (0, -1/2)$. This is also an example showing that Theorem 2 does not hold if we require $V \subset B$ instead of $\|v_i\| = 1$ for all i.

Note that in these theorems n has to be odd. Indeed, let w_1 and w_2 be two antipodal unit vectors. Set $n = 2k$, $v_1 = \dots = v_k = w_1$ and $v_{k+1} = \dots = v_n = w_2$. The conditions of Theorems 1 and 2 are satisfied (except that n is even now) but $\|v_1 + v_2 + ... + v_n\| = 0$. A minor modification of this example shows that n has to be odd in Theorem 3 as well. Namely, let the segment $[z_1, z_2]$ be a Euclidean diameter of B, and choose w_1, w_2 very close to z_1, z_2 so that $w_1 + w_2$ has norm $< 1/k$ and is orthogonal to z_1. This is clearly possible. Then with $n = 2k$, $v_1 = \dots = v_k = w_1$ and $v_{k+1} = \dots = v_n = w_2$ the conditions of Theorem 3 are satisfied but $\sum_1^n v_i \in B$.