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On the images of Lie polynomials evaluated on Lie algebras [☆]

Benjamin E. Anzis [a], Zachary M. Emrich [b,*], Kaavya G. Valiveti [c]

[a] *University of Idaho, United States*
[b] *Kent State University, United States*
[c] *University of California, Berkeley, United States*

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ABSTRACT

We describe the images of multilinear Lie polynomials of degrees 3 and 4 evaluated on $\mathfrak{su}(n)$ and $\mathfrak{so}(n)$.

1. Introduction

Recently there has been much interest [4,7,9,12] in a long-standing conjecture attributed to Lvov and Kaplansky regarding the images of multilinear polynomials (cf. [7]): "Let f be a multilinear polynomial over a field F. Then the set of values of f on the matrix algebra $M_n(F)$ is a vector space". Classical results of Shoda [10] and of Albert

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* Corresponding author.
E-mail address: zemrich@kent.edu (Z.M. Emrich).

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and Muckenhoupt [1] prove the conjecture for multilinear Lie polynomials of degree 2, and results of Špenko prove it for multilinear Lie polynomials of degrees 3 and 4.

A natural variation of the Lvov–Kaplansky conjecture is "Let f be a multilinear Lie polynomial over a field F. Then the set of values of f on classical Lie algebras is a vector space". For classical Lie algebras, Brown [3] proved an analogue of Shoda's result. In this paper, we prove this conjecture for multilinear Lie polynomials of degrees 3 and 4 evaluated on the Lie algebras $\mathfrak{su}(n)$ of traceless skew-Hermitian matrices and $\mathfrak{so}(n)$ of skew-symmetric matrices. Additionally, we prove the conjecture for degree 3 multilinear Lie polynomials evaluated on the Lie algebra $\mathfrak{sp}(2n)$ of Hamiltonian matrices.

The proofs for degree 3 are relatively straightforward. In the case of degree 4, we followed Špenko's ideas; however, to cover some special cases her result relied on [11, Theorem 1], which is not applicable in our situation. For $\mathfrak{su}(n)$, we use the well-known fact that any skew-Hermitian matrix is unitarily similar to a diagonal skew-Hermitian matrix. For $\mathfrak{so}(n)$, we rely on a highly nontrivial result on normal forms for skew-symmetric matrices by Djokovic, Rietsch, and Zhao [5].

All results in this paper are presented using the language of linear algebra to make the results accessible to a wider audience.

2. Preliminaries

Definition 1. Let $M, N \in M_n(\mathbb{C})$. Then, the commutator of M and N is

$$[M, N] = MN - NM.$$

We now introduce notation for the Lie algebras that we consider. Here $\mathfrak{su}(n)$ denotes the (real) Lie algebra of traceless $n \times n$ skew-Hermitian matrices over $\mathbb{C}$ and $\mathfrak{so}(n)$ the (complex) Lie algebra of $n \times n$ skew-symmetric matrices over $\mathbb{C}$.

We denote the Cartan subalgebra

$$\left\{ D \in \mathfrak{su}(n) \;\middle|\; D = \operatorname{diag}\left(\lambda_1, \lambda_2, \ldots, \lambda_{n-1}, -\sum_{k=1}^{n-1} \lambda_k \right) \right\}$$

of $\mathfrak{su}(n)$ by $H(\mathfrak{su}(n))$ and the set of zero-diagonal elements of $\mathfrak{su}(n)$ by $E(\mathfrak{su}(n))$. Note that $\mathfrak{su}(n) = H(\mathfrak{su}(n)) \oplus E(\mathfrak{su}(n))$.

We denote by $H(\mathfrak{so}(n))$ the Cartan subalgebra of $\mathfrak{so}(n)$ consisting of all skew-symmetric block diagonal matrices; if $n = 2k$, these matrices are of the form