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On essential spectra of singular linear Hamiltonian systems

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The paper is concerned with essential spectra of singular linear Hamiltonian systems of arbitrary order with arbitrary equal defect indices. Several sufficient conditions for the essential spectral points are given in terms of the number of linearly independent square integrable solutions of the corresponding Hamiltonian system, and a sufficient and necessary condition for the essential spectral points is obtained for Hamiltonian systems of even-order with one singular endpoint. An advantage of these results is that one can determine the essential spectral points of Hamiltonian systems by the information of solutions obtained by numerous tools available in the fundamental theory of differential equations. In addition, two illustrative examples are provided to show how to get some information about the essential spectrum by our results.

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1. Introduction

Consider the following linear Hamiltonian system:

$$\mathcal{L}(y)(t) := Jy'(t) - P(t)y(t) = \lambda W(t)y(t), \quad t \in (a,b), \tag{1.1$_\lambda$}$$

where $-\infty \le a < b \le +\infty$; $W(t)$ and $P(t)$ are $m \times m$ Hermitian matrices and locally integrable on (a,b); J is an $m \times m$ constant nonsingular matrix satisfying $J^* = -J$ (J^* denotes the complex conjugate transpose of J); $W(t) \ge 0$ is a weight function; and λ is a complex parameter. The formal operator $\mathcal{L}$ or (1.1$_\lambda$) is called regular at a if $a > -\infty$, and the above assumptions on the coefficients are satisfied in $[a,b)$ instead of (a,b). Otherwise, $\mathcal{L}$ or (1.1$_\lambda$) is called singular at a. The corresponding concepts for the endpoint b can be defined similarly.

Introduce the following space:

$$L^2_W(I) := \left\{ \text{measurable } f : I \to \mathbf{C}^m : \int_I f^*(t)W(t)f(t)\mathrm{d}t < +\infty \right\}$$

with inner product

$$\langle f, g\rangle = \int_I g^*(t)W(t)f(t)\mathrm{d}t,$$

where $I \subset (a,b)$ is an interval and $W(t)$ is given in (1.1$_\lambda$). Set $\|f\| = \langle f, f\rangle^{1/2}$ for $f \in L^2_W(I)$. We remark that if W is singular, $L^2_W(I)$ is a quotient space in the sense that $y = z$ if $\|y - z\| = 0$. In this case, $L^2_W(I)$ is a Hilbert space. Here, we also remark that for a scalar function $w > 0$, $L^2_w(I)$ consists of all weighted square integrable scalar functions on I, equipped with the inner product $\langle f, g\rangle = \int_I \bar{g}(t)w(t)f(t)\mathrm{d}t$ for $f, g \in L^2_w(I)$, where $\bar{g}(t)$ is the complex conjugate of $g(t)$.

Throughout the paper, in order to ensure the Hamiltonian operators generated by (1.1$_\lambda$) to be single-valued, it is always assumed that

(A) *For each pair of $a \le a' < b' \le b$, if y satisfies $Jy' - Py = Wf$ and $Wy = 0$ for $a' < t < b'$ for some $f \in L^2_W(a', b')$, then $y = 0$ for $a' < t < b'$.*

A similar assumption was used by Krall in [18,19]. Assumption (A) implies the Atkinson definiteness condition (see [1, p. 253]): for all $\lambda \in \mathbf{C}$ and for all the nontrivial solutions $y(t)$ of system (1.1$_\lambda$), the following inequality always holds:

$$\int_{a'}^{b'} y^*(s)W(s)y(s)\mathrm{d}s > 0, \quad a \le a' < b' \le b. \tag{1.2}$$