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# Recessive solutions for nonoscillatory discrete symplectic systems ☆

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ABSTRACT

In this paper we introduce a new concept of a recessive solution for discrete symplectic systems, which does not require any eventual controllability assumption. We prove that the existence of a recessive solution is equivalent to the nonoscillation of the system and that recessive solutions can have any rank between explicitly given lower and upper bounds. The smallest rank corresponds to the minimal recessive solution, which is unique up to a right nonsingular multiple, while the largest rank yields the traditional maximal recessive solution. We also present a method for constructing some (but not all) recessive solutions having a block diagonal structure from systems in lower dimension. Our results are new even for special discrete symplectic systems, such as for even order Sturm–Liouville difference equations and linear Hamiltonian difference systems.



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## 1. Introduction

In this paper we develop a general theory of the recessive solutions (also called the principal solutions at infinity or minimal solutions or distinguished solutions) for discrete symplectic systems

$$x_{k+1} = \mathcal{A}_k x_k + \mathcal{B}_k u_k, \qquad u_{k+1} = \mathcal{C}_k x_k + \mathcal{D}_k u_k, \quad k \in [0,\infty)_{\mathbb{Z}}, \tag{S}$$

where $\mathcal{A}_k$, $\mathcal{B}_k$, $\mathcal{C}_k$, $\mathcal{D}_k$ are real $n \times n$ matrices such that the transition matrix in system (S) is symplectic. More precisely, in the notation

$$z_{k+1} = \mathcal{S}_k z_k, \quad k \in [0,\infty)_{\mathbb{Z}}, \quad z_k = \begin{pmatrix} x_k \\ u_k \end{pmatrix}, \qquad \mathcal{S}_k = \begin{pmatrix} \mathcal{A}_k & \mathcal{B}_k \\ \mathcal{C}_k & \mathcal{D}_k \end{pmatrix}, \tag{1.1}$$

we require that the $2n \times 2n$ coefficient matrices $\mathcal{S}_k$ are symplectic, i.e., $\mathcal{S}_k^T \mathcal{J} \mathcal{S}_k = \mathcal{J}$ for all $k \in [0,\infty)_{\mathbb{Z}}$. Here $\mathcal{J} := \left(\begin{smallmatrix} 0 & I \\ -I & 0 \end{smallmatrix}\right)$ is the canonical skew-symmetric matrix of dimension $2n$ and $[N,\infty)_{\mathbb{Z}} := [N,\infty) \cap \mathbb{Z}$. According to [20, p. 5], a conjoined basis $(\hat{X}, \hat{U})$ of (S) is recessive at infinity if $\hat{X}_k$ is invertible for all $k \geq N$ for some $N \in [0,\infty)_{\mathbb{Z}}$ and

$$\lim_{k\to\infty} \hat{S}_k^{-1} = 0, \qquad \hat{S}_k := \sum_{j=N}^{k-1} \hat{X}_{j+1}^{-1} \mathcal{B}_j \hat{X}_j^{T-1}, \tag{1.2}$$

see also [26, p. 211]. This definition of a recessive solution covers the corresponding notion for the second order Sturm–Liouville difference equations $\Delta(r_k \Delta y_k) + q_k y_{k+1} = 0$, which was introduced by Hartman in [35] by the properties $\hat{y}_k \neq 0$ for large $k$ and

$$\sum_{k=N}^{\infty} \frac{1}{r_k \hat{y}_k \hat{y}_{k+1}} = \infty,$$

as well as it covers a recessive solution for higher order difference equations [35], three term recurrences [5,6], and linear Hamiltonian difference systems [10,31]. Recessive solutions for all these types of difference equations turned out to be an important tool for the investigation of their oscillation and spectral properties, see for example [1–4,11,13, 16–19,21–23,26,32,33,36,46,48,57]. An alternative definition of a recessive solution of (S) from [6, p. 115] based on a limit property is discussed in Remark 5.9.

The existence of a recessive solution of (S) is closely related with the nonoscillation of system (S), see Section 3 for more details, and with the concept of the eventual controllability of (S). More precisely, by [10] system (S) is called eventually controllable if there exists $N \in [0,\infty)_{\mathbb{Z}}$ such that if a solution $(x,u)$ of (S) satisfies $x_k = 0$ on a subinterval of $[N,\infty)_{\mathbb{Z}}$ with at least two points, then $(x,u) \equiv (0,0)$ on this subinterval, and hence $(x,u) \equiv (0,0)$ on $[0,\infty)_{\mathbb{Z}}$ by the uniqueness of solutions. The nonoscillation and eventual controllability of (S) imply that every conjoined basis $(X,U)$ of (S) has $X_k$