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Positive reduction from spectra

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ABSTRACT

We study the problem of whether all bipartite quantum states having a prescribed spectrum remain positive under the reduction map applied to one subsystem. We provide necessary and sufficient conditions, in the form of a family of linear inequalities, which the spectrum has to verify. Our conditions become explicit when one of the two subsystems is a qubit, as well as for further sets of states. Finally, we introduce a family of simple entanglement criteria for spectra, closely related to the reduction and positive partial transpose criteria, which also provide new insight into the set of spectra that guarantee separability or positivity of the partial transpose.

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1. Introduction

One of the most studied problems in quantum information theory is to find methods to decide whether a given quantum state is separable or entangled [14]. We recall that a quantum state $\rho \in M_n(\mathbb{C}) \otimes M_k(\mathbb{C})$ (here $M_n(\mathbb{C})$ denotes the space of all $n \times n$ complex matrices) is called *separable* [25] if it can be written as

$$\rho = \sum_i p_i e_i e_i^* \otimes f_i f_i^*$$

with $p_i \geq 0$, $\sum_i p_i = 1$, $e_i \in \mathbb{C}^n$, $f_i \in \mathbb{C}^k$ (throughout the paper we will identify states with their density matrices). States which are not separable are called *entangled*. Note that the set of separable states (SEP) is a convex subset of the convex set of all states. The extremal points of SEP are the pure product states, i.e. tensor products of one-dimensional projectors.

The separability problem has been proved to be *NP*-hard [8]. It can be mathematically related to positive maps on C^*-algebras since a quantum state $\rho \in M_n(\mathbb{C}) \otimes M_k(\mathbb{C})$ is separable if and only if $(\mathrm{id}_n \otimes P)(\rho)$ is positive-semidefinite for all positive maps $P : M_k(\mathbb{C}) \to M_m(\mathbb{C})$ and all positive integers $m \in \mathbb{N}$, where id_n is the identity map on some matrix algebra with appropriate dimension (here, n) [13]. Thus, each fixed positive map gives a necessary condition for separability. For example, the *positive partial transpose* (PPT) criterion corresponds to the choice $P = \Theta$, where Θ denotes the transposition map on $M_k(\mathbb{C})$. Moreover, the PPT criterion is also sufficient for $nk \leq 6$ [26,13], but this equivalence is wrong in higher dimensions.

An alternative choice of the positive map P is the *reduction map*

$$R : M_k(\mathbb{C}) \to M_k(\mathbb{C}), \quad R(X) := I_k \cdot \mathrm{Tr}[X] - X,$$

and the corresponding separability test is called *reduction* (RED) *criterion* [5,12]. The reduction criterion is weaker than the PPT criterion: if a state violates the reduction criterion, then it also violates the PPT criterion [12]. Conversely, there exist states (some entangled Werner states [25]) which satisfy the reduction criterion but violate the PPT criterion. On the other hand, the two criteria are equivalent if the subsystem on which the reduction map is applied is a qubit [5]. The importance of the reduction criterion stems from its connection to entanglement distillation [12]: any state which violates the reduction criterion is distillable. Recall that a bipartite entangled state is distillable if a pure maximally entangled state can be obtained arbitrarily closely, by local quantum operations and classical communication, from many copies of that state.

A possible approach to the separability problem is to study *absolutely separable states* (ASEP), i.e. states that remain separable under any global unitary transformation [18]. Since absolute separability is a purely spectral property, the problem is to find conditions on the spectrum that characterize absolutely separable states, i.e. to find constrains on the eigenvalues of a state ρ guaranteeing that ρ is separable with respect to any