Linear Algebra and its Applications 469 (2015) 305–334

Contents lists available at ScienceDirect

Linear Algebra and its Applications

www.elsevier.com/locate/laa

Change of the congruence canonical form of 2-by-2 and 3-by-3 matrices under perturbations and bundles of matrices under congruence

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ARTICLE INFO

Article history:
Received 19 January 2013
Accepted 6 November 2014
Available online 8 December 2014
Submitted by C. Mehl

MSC:
15A21
15A63

Keywords:
Closure graph
Congruence canonical form
Congruence class
Bundle
Perturbation

ABSTRACT

We construct the Hasse diagrams G_2 and G_3 for the closure ordering on the sets of congruence classes of 2×2 and 3×3 complex matrices. In other words, we construct two directed graphs whose vertices are 2×2 or, respectively, 3×3 canonical matrices under congruence, and there is a directed path from A to B if and only if A can be transformed by an arbitrarily small perturbation to a matrix that is congruent to B.

A bundle of matrices under congruence is defined as a set of square matrices A for which the pencils $A + \lambda A^T$ belong to the same bundle under strict equivalence. In support of this definition, we show that all matrices in a congruence bundle of 2×2 or 3×3 matrices have the same properties with respect to perturbations. We construct the Hasse diagrams G_2^{B} and G_3^{B} for the closure ordering on the sets of congruence bundles

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http://dx.doi.org/10.1016/j.laa.2014.11.004

of 2×2 and, respectively, 3×3 matrices. We find the isometry groups of 2×2 and 3×3 congruence canonical matrices.

1. Introduction

We study how small perturbations of a 2×2 or 3×3 complex matrix can change its congruence canonical form.

Two complex matrices A and B are said to be *congruent* if $S^TAS = B$ for a nonsingular S. This is an equivalence relation; its equivalence classes are called *congruence classes*. In Section 2 we construct the closure graphs G_2 and G_3, which are defined for any natural n as follows.

Definition 1.1. The *closure graph* G_n for congruence classes of $n\times n$ complex matrices is the directed graph in which each vertex v represents in a one-to-one manner a congruence class C_v of $n\times n$ matrices, and there is a directed path from a vertex v to a vertex w if and only if one (and hence each) matrix from C_v can be transformed to a matrix from C_w by an arbitrarily small perturbation.

The graph G_n is the Hasse diagram of the set of congruence classes of $n\times n$ matrices with the following partial order: $a \preccurlyeq b$ if a is contained in the closure of b. Thus, the graph G_n shows how the congruence classes relate to each other in the affine space of $n\times n$ matrices.

Since each $n\times n$ matrix is uniquely represented in the form $P+Q$ in which P is symmetric and Q is skew-symmetric, G_n is also the closure graph for congruence classes of $n\times n$ symmetric/skew-symmetric matrix pencils $P+\lambda Q$.

Each congruence class contains exactly one canonical matrix for congruence, and so it is convenient to represent the congruence classes by their canonical matrices. We use the congruence canonical matrices A_{can} constructed by Horn and Sergeichuk [21]. We also use the *miniversal deformation* of A_{can} given by Dmytryshyn, Futorny, and Sergeichuk [5]; that is, a simple normal form to which all matrices close to A_{can} can be reduced by congruence transformations that smoothly depend on their entries.

The closure graph for *congruence classes of 2×2 complex matrices was constructed by Futorny, Klimenko, and Sergeichuk [17].

Unlike perturbations of matrices under congruence and *congruence, perturbations of matrices under similarity and of matrix pencils have been much studied. For a given matrix A, Boer and Thijsse [4] and, independently, Markus and Parilis [30] described the set of all Jordan canonical matrices J such that in each neighborhood of A there exists a matrix whose Jordan canonical form is J. Their description was extended to Kronecker's canonical forms of pencils by Pokrzywa [32].