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Critical loci for projective reconstruction from multiple views in higher dimension: A comprehensive theoretical approach

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A B S T R A C T

A class of determinantal varieties, arising as critical loci for a natural generalization of a classical problem in computer vision, is introduced. Their ideals are investigated. These varieties are the critical loci for projective reconstruction from multiple views in higher dimension and, under suitable genericity assumptions, turn out to be hypersurfaces of degree r in $\mathbb{P}^{2r-1}$ or varieties of codimension 2 and degree $\frac{(r+2)(r+1)}{2}$ in $\mathbb{P}^{2r}$.

1. Introduction

Linear projections from $\mathbb{P}^3$ to $\mathbb{P}^2$ can naturally be utilized to model images of static three-dimensional scenes, assuming that cameras are very simply thought of as pinholes.

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More generally, linear projections from $\mathbb{P}^k$ to $\mathbb{P}^2$, and even from $\mathbb{P}^k$ to $\mathbb{P}^m$, $m \geq 3$, were also found helpful by the computer vision community in modeling images of particular dynamic and segmented scenes, [22,10,14,8,9,20,21]. A classical problem in computer vision is *reconstruction*: given multiple images of an unknown scene, taken from unknown cameras, try to reconstruct the positions of the cameras and of the scene points. It is not difficult to see that sufficiently many images, and sufficiently many sets of corresponding points in the given images, chosen as images of the same set of test points in space, should allow for a successful projective reconstruction. Indeed the vision community has over the years produced quite sophisticated algorithms to perform reconstruction in many different settings. On the other hand, it is also relatively easy to notice that even in the classical setup of two projections from $\mathbb{P}^3$ to $\mathbb{P}^2$ there are set of *critical* points which prevent projective reconstruction: i.e. non-projectively equivalent pairs of sets of scene points and of cameras that produce the same images in the view planes. Critical loci in $\mathbb{P}^3$ have been the object of interest for several authors for quite some time. In the case of one view, the classical result of the criticality of a twisted cubic curve goes back to [6]. Quadric surfaces were shown to be critical for two views in [17,18]. Contributions in the case of three or more views are found in [11,15], and [19]. A comprehensive, detailed analysis both in the case of two and in the case of multiple views was conducted in [12].

In the case of projections from higher dimensional $\mathbb{P}^k$ to $\mathbb{P}^2$, critical loci in the case of one view were theoretically described in [5]. Experimental evidence of the instability of reconstruction near such critical loci was given in [1]. A general framework to understand critical loci for projective reconstruction from multiple views in higher dimensions was given in [2] and [4].

The framework identified in the latter two papers clearly showed that critical loci are special determinantal varieties. The main goal of this work is to study critical loci as zero-sets of suitable ideals.

As a by product of revisiting the framework we also properly reset the problem in a fully projective context and, as a result, we are able to significantly simplify and enhance results obtained in previous works. Critical loci for projective reconstruction from the minimum number n of necessary views in $\mathbb{P}^k$ turn out to be either hypersurfaces of degree $\frac{k+1}{2} = n$, if the ambient space is odd dimensional, see Theorem 4.1, or special determinantal varieties of codimension 2 and degree $\frac{(k+4)(k+2)}{8}$ if the ambient space is even dimensional, see Section 5.

The paper is organized as follows: In Section 2 notations are fixed and a brief introduction to the general computer vision setting is offered for the convenience of the reader. Section 3 describes the general theoretical framework for critical configurations and critical loci, in a fully projective context. Sections 4 and 5 are dedicated, respectively, to the case of odd and even dimensional ambient spaces. The authors would like to thank the anonymous referee for suggesting a more explicit treatment of degenerate cases for the critical locus, and the consideration of group actions in Sections 4 and 5. The authors also thank Roberto Notari for helpful conversations on degenerate config-