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On the game-theoretic value of a linear transformation relative to a self-dual cone

M. Seetharama Gowda [a,*], G. Ravindran [b]

[a] *Department of Mathematics and Statistics, University of Maryland, Baltimore County, Baltimore, MD 21250, United States*
[b] *Indian Statistical Institute, Chennai, 600113, India*

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ABSTRACT

This paper is concerned with a generalization of the concept of value of a (zero-sum) matrix game. Given a finite dimensional real inner product space V with a self-dual cone K, an element e in the interior of K, and a linear transformation L, we define the value of L by

$$v(L) := \max_{x\in\Delta} \min_{y\in\Delta} \langle L(x), y\rangle = \min_{y\in\Delta} \max_{x\in\Delta} \langle L(x), y\rangle,$$

where $\Delta = \{x \in K : \langle x, e\rangle = 1\}$. This reduces to the classical value of a square matrix when $V = R^n$, $K = R^n_+$, and e is the vector of ones. In this paper, we extend some classical results of Kaplansky and Raghavan to this general setting. In addition, for a **Z**-transformation (which is a generalization of a **Z**-matrix), we relate the value with various properties such as the positive stable property, the **S**-property, etc. We apply these results to find the values of the Lyapunov transformation L_A and the Stein transformation S_A on the cone of $n \times n$ real symmetric positive semidefinite matrices.

* Corresponding author.
E-mail addresses: gowda@umbc.edu (M.S. Gowda), gravi@hotmail.com (G. Ravindran).

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1. Introduction

This paper is concerned with a generalization of the concept of value of a (zero-sum) matrix game. To explain, we consider an $n \times n$ real matrix A and the strategy set $X := \{x \in R^n_+ : \sum_1^n x_i = 1\}$, where R^n_+ denotes the nonnegative orthant in R^n. Then the value of A is given by

$$v(A) := \max_{x\in X} \min_{y\in X} \langle Ax, y\rangle = \min_{y\in X} \max_{x\in X} \langle Ax, y\rangle,$$

where $\langle Ax, y\rangle$ denotes the (usual) inner product between vectors Ax and y. Corresponding to this, there exist *optimal* strategies $\bar{x}, \bar{y} \in X$ such that

$$\langle Ax, \bar{y}\rangle \le v(A) = \langle A\bar{x}, \bar{y}\rangle \le \langle A\bar{x}, y\rangle \quad \forall x, y \in X.$$

The concept of value of a matrix and its applications are classical and have been well studied and documented in the game theory literature; see, for example, [12,13]. Our motivation for the generalization comes from results of Kaplansky and Raghavan. In [11], Kaplansky defines a completely mixed (matrix) game as one in which $\bar{x} > 0$ and $\bar{y} > 0$ for every pair of optimal strategies $(\bar{x}, \bar{y})$. For such a game, Kaplansky proves the uniqueness of the optimal strategy pair. In [14], Raghavan shows that for a **Z**-matrix (which is a square matrix whose off-diagonal entries are all non-positive) the game is completely mixed when the value is positive, and relates the property of value being positive to a number of equivalent properties of the matrix such as the positive stable property, the **P**-property, etc. His result, in particular, says that for a **Z**-matrix A, the value is positive if and only if there exists an $\bar{x} \in R^n$ such that

$$\bar{x} > 0 \quad \text{and} \quad A\bar{x} > 0.$$

Inequalities of the above type also appear in the study of linear continuous and discrete dynamical systems: Given an $n \times n$ real matrix A, the continuous dynamical system $\frac{dx}{dt} + Ax(t) = 0$ is asymptotically stable on R^n (which means that any trajectory starting from an arbitrary point in R^n converges to the origin) if and only if there exists a real symmetric matrix $\overline{X}$ such that

$$\overline{X} > 0 \quad \text{and} \quad L_A(\overline{X}) > 0,$$

where $\overline{X} > 0$ means that $\overline{X}$ is positive definite, etc., and L_A denotes the so-called *Lyapunov transformation* defined on the space $\mathcal{S}^n$ of all $n \times n$ real symmetric matrices:

$$L_A(X) := AX + XA^T \quad \left(X \in \mathcal{S}^n\right).$$

Similarly, the discrete dynamical system $x(k+1) = Ax(k)$, $k = 0, 1, \ldots,$ is asymptotically stable on R^n if and only if there exists a real symmetric matrix $\overline{X}$ such that