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Asymptotics of eigenvalues of symmetric Toeplitz band matrices

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In this paper we obtain uniform asymptotic formulas for all eigenvalues of symmetric Toeplitz band matrices of large dimension. The entries of the matrices are assumed to be complex, that is, the matrices need not necessarily be self-adjoint. The formulas presented allow a detailed study of the structure of the eigenvalues and of their location in the complex plane, and to build an efficient algorithm for finding their numerical values for matrices of medium and high dimension.

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1. Introduction

The $n \times n$ matrix $T_n(a)$ generated by the function (symbol) a in L^1 on the complex unit circle $\mathbb{T}$ is defined by $T_n(a) = (a_{j-k})_{j,k=1}^n$ where a_ℓ is the ℓ-th Fourier coefficient of a,

$$a_\ell = \frac{1}{2\pi} \int_0^{2\pi} a(e^{ix}) e^{-i\ell x} dx \quad (\ell \in \mathbb{Z}).$$

The asymptotics of eigenvalues of $T_n(a)$ as $n \to \infty$ has attracted mathematicians and physicists for a century. The collective asymptotic behavior of the eigenvalues of Hermitian Toeplitz matrices is described by the first Szegö limit theorem; see [1–6]. In the Hermitian case extensive work has also been done on the extreme eigenvalues $T_n(a)$; see for example [7–14].

Results on the uniform asymptotics of all (extreme and inner) eigenvalues of Hermitian Toeplitz matrices were obtained only recently [15,16].

The case of non-Hermitian Toeplitz matrices is more complicated; see the papers [17–20] and the books [21,22].

In this paper we consider the problem of constructing higher-order asymptotic formulas for the eigenvalues of $T_n(a)$, denoted by $\lambda_{j,n}$ as $n \to \infty$ uniformly with $1 \le j \le n$, in the case that a is a symmetric Laurent polynomial with complex coefficients,

$$a(t) = \sum_{k=-r}^{r} a_k t^k, \quad a_k = a_{-k}, \ t \in \mathbb{T} \tag{1.1}$$

with $r \ge 1$ and $a_r \ne 0$. This polynomial is an even function with respect to the angular variable φ:

$$a(e^{i\varphi}) = a(e^{-i\varphi}), \quad \varphi \in [0, \pi]. \tag{1.2}$$

Consider the curve

$$\mathcal{R}(a) := a(\mathbb{T}).$$

Then by [19,20] it follows that $\mathcal{R}(a)$ is the limit set of the set of eigenvalues $\{\lambda_{j,n}\}_{j=1}^n$ of the operators $T_n(a)$, $n \to \infty$. Thus for all sufficiently large n, $\lambda_{j,n}$ is located in a small neighborhood of $\mathcal{R}(a)$. In this paper we significantly refine the results of these works by defining decreasing small areas such that each eigenvalue is located in one of them for large n. We present an iterative algorithm and an asymptotic formula for quickly calculating the eigenvalues and exploring their location in $\mathbb{C}$ relative to the curve $\mathcal{R}(a)$.

We use ideas and methods which were developed in [15,23]. Note, however, that complex-valued symbols, in contrast to the real-valued case of paper [15] cause a number of significant challenges.