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The Laplacian energy of threshold graphs and majorization

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ABSTRACT

We study the Laplacian energy of threshold graphs, inspired by the recent results of Vinagre, Del-Vecchio, Justo and Trevisan [22]. In particular, we compute the degree sequences of threshold graphs that maximize (or minimize) the Laplacian energy for a fixed number of vertices and edges. The analysis involves combinatorial methods using Ferrers diagrams and ideas from majorization theory. Some new inequalities for threshold degree sequences are obtained in this process. In the review process a referee pointed out that, recently and independently, Helmberg and Trevisan [14] obtained very similar results, and we discuss this connection.

1. Introduction

Let $L(G)$ denote the Laplacian matrix [5] of a graph G with n vertices. $L(G)$ is symmetric, positive semidefinite and singular, so it has real, nonnegative eigenvalues $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n = 0$. The *Laplacian energy* [12] of G is defined as

$$LE(G) = \sum_{i=1}^{n} \left| \mu_i - (2m/n) \right|$$

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and it measures how the Laplacian eigenvalues deviate from the average degree $2m/n$, see [22]. For a thorough treatment of spectral graph theory, we refer to the recent monograph [3]. The computation of Laplacian spectra is hard in general, but methods for computing such spectra, including finding bounds, have been discussed for certain classes of graphs, see [1,6,20] and the references given there. A *threshold graph* ([7], [11, Chapter 10]) is a graph that can be constructed by starting with a single vertex and successively adding a new vertex which is either isolated or dominating (i.e., it has an edge to all previous vertices). The number of dominating vertices is called the *trace* of the threshold graph. Threshold graphs arise is certain applications and have many interesting properties. In [22] one studied the maximum Laplacian energy of a class of threshold graphs, and it was shown that the maximum is attained for so-called pineapple graphs (see later). The purpose of the present paper is to extend and supplement those results, and we treat threshold graphs in general. Interestingly, our methods are very different from those of [22]; we use combinatorial arguments, with geometrical interpretation, based on integer partitions and ideas from majorization theory.

It was kindly pointed out by a referee that, in parallel to the present work, Christoph Helmberg and Vilmar Trevisan [14] had worked on the same kind of problem and obtained several similar results. We stress that our paper and [14] are completely independent works, and done more or less at the same time. As a part of this story Vilmar Trevisan contacted the present author at the ILAS 2013 meeting in Providence, and pointed out the recent paper [22] and mentioned an open problem concerning Laplacian energy (which is still open). The author is grateful to Vilmar Trevisan for this nice and interesting conversation which introduced this author to threshold graphs and Laplacian energy. Some of our results, concerning maximizers of Laplacian energy for fixed number of vertices and edges are similar to those found in [14]. Both papers rely on the analysis of Ferrers diagrams for these graphs, and fixing the trace in the analysis. This is a natural approach since the Laplacian eigenvalues are easily determined by the Ferrers diagrams. The present paper, however, uses a different approach linked to majorization of degree vectors and, related to the main result, we obtain some interesting inequalities for so-called k-maximal and k-minimal threshold degree vectors. On the other hand, in [14] one also solves the problem of maximizing the Laplacian energy among threshold graphs where only the number of vertices is fixed, and show that the pineapple maximizes the Laplacian energy in that situation.

We call a vector $x = (x_1, x_2, \dots, x_n)$ *monotone* if $x_1 \geq x_2 \geq \cdots \geq x_n$. Let $G = (V, E)$ be a graph with n vertices and m edges, and let $d = (d_1, d_2, \dots, d_n)$ be its monotone *degree vector*, i.e., d_i is the ith largest degree of a vertex in G. Then $\sum_{i=1}^n d_i = 2m$, so $d_1, d_2, \dots, d_n$ are the parts of an *integer partition* of $2m$. This integer partition may be represented by a Ferrers diagram where the ith degree is indicated by d_i boxes, drawn horizontally (where a vertical line indicates no boxes). For instance if $d = (4, 2, 1, 0)$ the Ferrers diagram is