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Multiplicative Lidskii's inequalities and optimal perturbations of frames ☆

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In this paper we study two design problems in frame theory: on the one hand, given a fixed finite frame $\mathcal{F} = \{f_j\}_{j\in\mathbb{I}_n}$ for $\mathbb{C}^d$ we compute those dual frames $\mathcal{G}$ of $\mathcal{F}$ that are optimal perturbations of the canonical dual frame for $\mathcal{F}$ under certain restrictions on the norms of the elements of $\mathcal{G}$. On the other hand, we compute those $V \cdot \mathcal{F} = \{Vf_j\}_{j\in\mathbb{I}_n}$ – for invertible operators V which are close to the identity – that are optimal perturbations of $\mathcal{F}$. That is, we compute the optimal perturbations of $\mathcal{F}$ among frames $\mathcal{G} = \{g_j\}_{j\in\mathbb{I}_n}$ that have the same linear relations as $\mathcal{F}$. In both cases, optimality is measured with respect to submajorization of the eigenvalues of the frame operators. Hence, our optimal designs are minimizers of a family of convex potentials that include the frame potential and the mean squared error. The key tool for these results is a multiplicative analogue of Lidskii's inequality in terms of log-majorization and a characterization of the case of equality.

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1. Introduction

A finite frame for $\mathbb{C}^d$ is a sequence $\mathcal{F} = \{f_j\}_{j\in\mathbb{I}_n}$ that spans $\mathbb{C}^d$, where $\mathbb{I}_n = \{1, \ldots, n\}$ (for a detailed exposition on frames and several recent research topics within this theory see [8,9] and the references therein). Given a frame $\mathcal{F} = \{f_j\}_{j\in\mathbb{I}_n}$, a sequence $\mathcal{G} = \{g_j\}_{j\in\mathbb{I}_n}$ is called a dual frame for $\mathcal{F}$ if for every $f \in \mathbb{C}^d$ the following reconstruction formulas hold:

$$f = \sum_{j\in\mathbb{I}_n} \langle f, g_j\rangle f_j \quad \text{and} \quad f = \sum_{j\in\mathbb{I}_n} \langle f, f_j\rangle g_j.$$

Hence, frames provide a (possibly redundant) linear-encoding scheme for vectors in $\mathbb{C}^d$.

Let $\mathcal{F} = \{f_j\}_{j\in\mathbb{I}_n}$ be a frame for $\mathbb{C}^d$ and let $\mathcal{D}(\mathcal{F})$ denote the set of dual frames for $\mathcal{F}$. There is a distinguished dual called the canonical dual of $\mathcal{F}$, denoted $\mathcal{F}^\# \in \mathcal{D}(\mathcal{F})$, which is a natural choice in several ways. But in case $n > d$ it is well known that $\mathcal{D}(\mathcal{F})$ has a rich structure (this last fact is one of the main advantages of frames over bases $\mathcal{B} = \{v_j\}_{i\in\mathbb{I}_d}$ for which $\mathcal{D}(\mathcal{B})$ becomes a singleton). Thus, in applied situations, the structure of $\mathcal{D}(\mathcal{F})$ can be exploited to obtain numerically stable encoding–decoding schemes derived from the dual pair $(\mathcal{F}, \mathcal{G})$, for some choice of dual frame $\mathcal{G} \in \mathcal{D}(\mathcal{F})$ beyond $\mathcal{F}^\#$; this is the starting point of the so-called (optimal) design problems for dual frames (see [13,15,17–19]).

Another research topic in frame theory is the design of (optimal) stable configurations of vectors (frames) under certain restrictions. Typically, the stability of a frame $\mathcal{F}$ is measured in terms of the spread the eigenvalues of the positive semidefinite operator $S_\mathcal{F} = \sum_{j\in\mathbb{I}_n} f_j \otimes f_j$. One of the most important examples of such a measure is the frame potential of $\mathcal{F}$, denoted by $\mathrm{FP}(\mathcal{F})$, introduced in [5]; explicitly, for a sequence $\mathcal{F} = \{f_j\}_{j\in\mathbb{I}_n}$ then

$$\mathrm{FP}(\mathcal{F}) = \sum_{j,k\in\mathbb{I}_n} \left|\langle f_j, f_k\rangle\right|^2 = \mathrm{tr}\left(S_\mathcal{F}^2\right).$$

In [5,7] it is shown that minimizers of the frame potential, within convenient sets of frames, have many nice structural features and are optimal in several ways. Recently, there has also been interest in the so-called mean squared error of $\mathcal{F}$, denoted $\mathrm{MSE}(\mathcal{F})$, given by $\mathrm{MSE}(\mathcal{F}) = \mathrm{tr}(S_\mathcal{F}^{-1})$ (see [11,16,20]).

It turns out that there is a structural measure of optimality, called sub-majorization, that allows to deal with both the frame potential and the mean squared error. This pre-order relation, defined between eigenvalues of frame operators, has proved useful in explaining the structure of minimizers of convex potentials (see [16]). Sub-majorization has also been useful in obtaining the structure of optimal vector configurations as well (see [19,21]). In turn, sub-majorization relations imply a family of tracial inequalities in terms of convex functions, that contain the frame potential and mean squared errors. We point out that these tracial inequalities have interest in their own right and collectively characterize sub-majorization.