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Hermitian unitary matrices with modular permutation symmetry

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We study Hermitian unitary matrices $\mathcal{S} \in \mathbb{C}^{n,n}$ with the following property: There exist $r \geq 0$ and $t > 0$ such that the entries of $\mathcal{S}$ satisfy $|\mathcal{S}_{jj}| = r$ and $|\mathcal{S}_{jk}| = t$ for all $j,k = 1,\dots,n$, $j \neq k$. We derive necessary conditions on the ratio $d := r/t$ and show that these conditions are very restrictive except for the case when n is even and the sum of the diagonal elements of $\mathcal{S}$ is zero. Examples of families of matrices $\mathcal{S}$ are constructed for d belonging to certain intervals. The case of real matrices $\mathcal{S}$ is examined in more detail. It is demonstrated that a real $\mathcal{S}$ can exist only for $d = \frac{n}{2} - 1$, or for n even and $\frac{n}{2} + d \equiv 1 \pmod 2$. We provide a detailed description of the structure of real $\mathcal{S}$ with $d \geq \frac{n}{4} - \frac{3}{2}$, and derive a sufficient and necessary condition of its existence in terms of the existence of certain symmetric (v,k,λ)-designs. We prove that there exists no real $\mathcal{S}$ with $d \in (\frac{n}{6} - 1, \frac{n}{4} - \frac{3}{2})$. A parametrization of Hermitian unitary matrices is also proposed, and its generalization to general unitary matrices is given. At the end of the paper, the role of the studied matrices in quantum mechanics on graphs is briefly explained.

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1. Introduction

Unitary matrices with various special properties emerge in a wide scale of applications in physics and in the engineering, and at the same time they constantly attract the attention of pure mathematicians. One of the most fascinating and longest-standing problems in mathematics is the Hadamard conjecture: *If n is a multiple of 4, then there exists an $n \times n$ matrix H with entries from $\{-1, 1\}$ such that $HH^T = nI$.* Although the conjecture is believed to be true, no proof has yet been found. The matrix H with these properties is called Hadamard matrix of order n, and it is just a multiple of an orthogonal matrix having all the entries of the same moduli. Hadamard matrices have numerous practical applications in coding, cryptography, signal processing, artificial neural networks and many other fields, see, e.g., the monograph [1].

A similar problem is related to the existence of so-called conference matrices. A conference matrix of order n is an $n \times n$ matrix C with 0 on the diagonal and ± 1 off the diagonal such that $CC^T = (n-1)I$. Matrices of this type are important for example in telephony [2] and in statistics [3], but as in the case of Hadamard matrices, there is still no definite characterization of orders n for which a conference matrix exists.

Note that both Hadamard and conference matrices have these two properties:

(P1) they are multiples of orthogonal matrices;
(P2) all their off-diagonal entries are of the same moduli, and also all their diagonal entries are of the same moduli.

These properties can serve as an inspiration to generalize Hadamard and conference matrices to the whole set of matrices satisfying (P1) and (P2). A subclass fulfilling a certain additional condition, namely the class of matrices with constant diagonal, has been studied in [4,5].

Both Hadamard and conference matrices are by definition real, but they can be naturally generalized to complex ones by allowing their entries to take any values from the unit circle instead of $\{1, -1\}$. Complex Hadamard and conference matrices and their properties are nowadays widely studied as well, see, e.g., [1,6]. This fact may serve as another inspiration for generalizations: Examine all unitary matrices satisfying (P2).

The subject to be discussed in this paper is close to the aforementioned generalization. We will study complex unitary matrices satisfying (P2) that are also Hermitian. Our aim is to examine their existence and their properties, and perhaps to motivate a more extensive study of them, as they play an important role in the quantum mechanics on graphs (we will devote Section 7 at the end of the paper to a more detailed explanation). Since the real matrices of this type are for many reasons interesting, we will focus on the real case in a separate section. Another purpose of the paper is to propose a parametrization of unitary matrices, with a particular accent put on their Hermitian subset.