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Majorization and refined Jensen–Mercer type inequalities for self-adjoint operators

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In this paper, Jensen–Mercer's inequality is generalized by applying the method of pre-majorization used for comparing two tuples of self-adjoint operators. A general result in a matrix setting is established. Special cases of the main theorem are studied to recover other inequalities of Mercer type.

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1. Introduction and summary

Mercer [11] established a variant of Jensen's inequality as follows.

Theorem A. *(See [11, Theorem 1.2].) Let f be a real convex function on an interval $[a_1, a_2]$, $a_1 < a_2$, and $a_1 \le b_i \le a_2$ for $i = 1, \ldots, n$. Then*

$$f\left(a_1 + a_2 - \sum_{i=1}^{n} w_i b_i\right) \le f(a_1) + f(a_2) - \sum_{i=1}^{n} w_i f(b_i), \tag{1}$$

where $\sum_{i=1}^{n} w_i = 1$ with $w_i > 0$.

Relation (1) is referred as *Jensen–Mercer's inequality.*

An m-tuple $\mathbf{b} = (b_1, \ldots, b_m) \in \mathbf{R}^m$ is said to be *majorized* by an m-tuple $\mathbf{a} = (a_1, \ldots, a_m) \in \mathbf{R}^m$, written as $\mathbf{b} \prec \mathbf{a}$ if

$$\sum_{l=1}^{k} b_{[l]} \le \sum_{l=1}^{k} a_{[l]} \quad \text{for } k = 1, \ldots, m, \quad \text{and} \quad \sum_{l=1}^{m} b_j = \sum_{l=1}^{m} a_j,$$

where $a_{[1]} \ge \cdots \ge a_{[m]}$ and $b_{[1]} \ge \cdots \ge b_{[m]}$ are the entries of $\mathbf{a}$ and $\mathbf{b}$, respectively, in nonincreasing order [9, p. 8].

A majorization approach to Jensen–Mercer's inequality was shown in [14].

Theorem B. *(See [14, Theorem 2.1].) Let $f : J \to \mathbf{R}$ be a continuous convex function on interval $J \subseteq \mathbf{R}$. Suppose $\mathbf{a} = (a_1, \ldots, a_m)$ with $a_l \in J$, $l = 1, \ldots, m$, and $\mathbf{B} = (b_{ij})$ is a real $n \times m$ matrix such that $b_{ij} \in J$ for $i = 1, \ldots, n$, $j = 1, \ldots, m$.*

If $\mathbf{a}$ majorizes each row of $\mathbf{B}$, that is

$$\mathbf{b}_{i\cdot} = (b_{i1}, \ldots, b_{im}) \prec (a_1, \ldots, a_m) = \mathbf{a} \quad \text{for each } i = 1, \ldots, n,$$

then

$$f\left(\sum_{l=1}^{m} a_l - \sum_{j=1}^{m-1} \sum_{i=1}^{n} w_i b_{ij}\right) \le \sum_{l=1}^{m} f(a_l) - \sum_{j=1}^{m-1} \sum_{i=1}^{n} w_i f(b_{ij}), \tag{2}$$

where $\sum_{i=1}^{n} w_i = 1$ with $w_i \ge 0$.

The symbol $\mathbf{B}(\mathcal{H})$ stands for the linear space of all bounded linear operators on a Hilbert space $\mathcal{H}$. For selfadjoint operators $A, B \in \mathbf{B}(\mathcal{H})$, we write $A \le B$ if $B - A$ is positive, i.e., $\langle Ah, h\rangle \le \langle Bh, h\rangle$ for all $h \in \mathcal{H}$. A linear mapping $\Phi : \mathbf{B}(\mathcal{H}) \to \mathbf{B}(\mathcal{K})$ is said to be a *positive linear map* if $\Phi(A) \le \Phi(B)$ for all $A, B \in \mathbf{B}(\mathcal{H})$ such that $A \le B$.

A continuous function $f : J \to \mathbf{R}$ defined on an interval $J \subseteq \mathbf{R}$ is said to be *operator convex* if $f(\lambda A + (1 - \lambda)B) \le \lambda f(A) + (1 - \lambda)f(B)$ for any $\lambda \in [0, 1]$ and all self-adjoint operators A, B with spectra in J.