

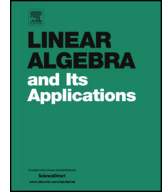


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# Majorization and refined Jensen–Mercer type inequalities for self-adjoint operators



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### ABSTRACT

In this paper, Jensen–Mercer's inequality is generalized by applying the method of pre-majorization used for comparing two tuples of self-adjoint operators. A general result in a matrix setting is established. Special cases of the main theorem are studied to recover other inequalities of Mercer type.

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## 1. Introduction and summary

Mercer [11] established a variant of Jensen's inequality as follows.

**Theorem A.** (See [11, Theorem 1.2].) Let  $f$  be a real convex function on an interval  $[a_1, a_2]$ ,  $a_1 < a_2$ , and  $a_1 \leq b_i \leq a_2$  for  $i = 1, \dots, n$ . Then

$$f\left(a_1 + a_2 - \sum_{i=1}^n w_i b_i\right) \leq f(a_1) + f(a_2) - \sum_{i=1}^n w_i f(b_i), \quad (1)$$

where  $\sum_{i=1}^n w_i = 1$  with  $w_i > 0$ .

Relation (1) is referred as *Jensen–Mercer's inequality*.

An  $m$ -tuple  $\mathbf{b} = (b_1, \dots, b_m) \in \mathbf{R}^m$  is said to be *majorized* by an  $m$ -tuple  $\mathbf{a} = (a_1, \dots, a_m) \in \mathbf{R}^m$ , written as  $\mathbf{b} \prec \mathbf{a}$  if

$$\sum_{l=1}^k b_{[l]} \leq \sum_{l=1}^k a_{[l]} \quad \text{for } k = 1, \dots, m, \quad \text{and} \quad \sum_{l=1}^m b_j = \sum_{l=1}^m a_j,$$

where  $a_{[1]} \geq \dots \geq a_{[m]}$  and  $b_{[1]} \geq \dots \geq b_{[m]}$  are the entries of  $\mathbf{a}$  and  $\mathbf{b}$ , respectively, in nonincreasing order [9, p. 8].

A majorization approach to Jensen–Mercer's inequality was shown in [14].

**Theorem B.** (See [14, Theorem 2.1].) Let  $f : J \rightarrow \mathbf{R}$  be a continuous convex function on interval  $J \subseteq \mathbf{R}$ . Suppose  $\mathbf{a} = (a_1, \dots, a_m)$  with  $a_l \in J$ ,  $l = 1, \dots, m$ , and  $\mathbf{B} = (b_{ij})$  is a real  $n \times m$  matrix such that  $b_{ij} \in J$  for  $i = 1, \dots, n$ ,  $j = 1, \dots, m$ .

If  $\mathbf{a}$  majorizes each row of  $\mathbf{B}$ , that is

$$\mathbf{b}_{i\cdot} = (b_{i1}, \dots, b_{im}) \prec (a_1, \dots, a_m) = \mathbf{a} \quad \text{for each } i = 1, \dots, n,$$

then

$$f\left(\sum_{l=1}^m a_l - \sum_{j=1}^{m-1} \sum_{i=1}^n w_i b_{ij}\right) \leq \sum_{l=1}^m f(a_l) - \sum_{j=1}^{m-1} \sum_{i=1}^n w_i f(b_{ij}), \quad (2)$$

where  $\sum_{i=1}^n w_i = 1$  with  $w_i \geq 0$ .

The symbol  $\mathbf{B}(\mathcal{H})$  stands for the linear space of all bounded linear operators on a Hilbert space  $\mathcal{H}$ . For selfadjoint operators  $A, B \in \mathbf{B}(\mathcal{H})$ , we write  $A \leq B$  if  $B - A$  is positive, i.e.,  $\langle Ah, h \rangle \leq \langle Bh, h \rangle$  for all  $h \in \mathcal{H}$ . A linear mapping  $\Phi : \mathbf{B}(\mathcal{H}) \rightarrow \mathbf{B}(\mathcal{K})$  is said to be a *positive linear map* if  $\Phi(A) \leq \Phi(B)$  for all  $A, B \in \mathbf{B}(\mathcal{H})$  such that  $A \leq B$ .

A continuous function  $f : J \rightarrow \mathbf{R}$  defined on an interval  $J \subseteq \mathbf{R}$  is said to be *operator convex* if  $f(\lambda A + (1 - \lambda)B) \leq \lambda f(A) + (1 - \lambda)f(B)$  for any  $\lambda \in [0, 1]$  and all self-adjoint operators  $A, B$  with spectra in  $J$ .

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