Linear Algebra and its Applications 467 (2015) 15–28

Contents lists available at ScienceDirect

Linear Algebra and its Applications

www.elsevier.com/locate/laa

Strong shift equivalence and positive doubly stochastic matrices

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ARTICLE INFO

Article history:
Received 16 July 2014
Accepted 29 October 2014
Available online 14 November 2014
Submitted by R. Brualdi

MSC:
15B48
37B10
15A21
15B51

Keywords:
Shift equivalence
Spectra
Stochastic matrices

ABSTRACT

We give sufficient conditions for a positive stochastic matrix to be similar and strong shift equivalent over $\mathbb{R}_+$ to a positive doubly stochastic matrix through matrices of the same size. We also prove that every positive stochastic matrix is strong shift equivalent over $\mathbb{R}_+$ to a positive doubly stochastic matrix. Consequently, the set of nonzero spectra of primitive stochastic matrices over $\mathbb{R}$ with positive trace and the set of nonzero spectra of positive doubly stochastic matrices over $\mathbb{R}$ are identical. We exhibit a class of 2×2 matrices, pairwise strong shift equivalent over $\mathbb{R}_+$ through 2×2 matrices, for which there is no uniform upper bound on the minimum lag of a strong shift equivalence through matrices of bounded size. In contrast, we show for any $n \times n$ primitive matrix of positive trace that the set of positive $n \times n$ matrices similar to it contains only finitely many SSE–$\mathbb{R}_+$ classes.

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http://dx.doi.org/10.1016/j.laa.2014.10.043

1. Introduction

Strong shift equivalence theory (for matrices over $\mathbb{Z}_+$) was introduced by R.F. Williams [17] as a tool for classifying shifts of finite type. Subsequently, strong shift equivalence over other semirings has been used for classification of other symbolic dynamical systems such as SFTs with Markov measure [16,15] and SFTs with a free finite group action [5].

Despite its good-looking definition, strong shift equivalence is still very difficult to fully understand. Williams also introduced a more tractable equivalence relation called shift equivalence and conjectured that shift equivalence and strong shift equivalence over $\mathbb{Z}_+$ are the same. The conjecture was proved false by K.H. Kim and F.W. Roush in 1992 (reducible case) [8] and 1997 (irreducible case) [9]. Although Williams' conjecture is false in general, the gap between shift equivalence and strong shift equivalence over $\mathbb{Z}_+$ remains mysterious. We study Williams' conjecture by relaxing the problem to the level of positive rational and real matrices, as in [12,11,10,4]. We expect that the Williams conjecture is true for positive rational (or real) matrices. This is the conjecture posed by Mike Boyle in [2, Conjecture 5.1]. Understanding this relation is a natural step toward understanding strong shift equivalence over $\mathbb{Z}_+$, and a natural matrix problem independently.

The purpose of this paper is twofold. Firstly, it gives a connection between stochastic matrices and doubly stochastic matrices via strong shift equivalence, in both local and global aspects. In Section 3, we give sufficient conditions for an $n \times n$ positive stochastic matrix P over a subfield $\mathbb{U}$ of $\mathbb{R}$ to be strong shift equivalent over $\mathbb{U}_+$ to a positive doubly stochastic matrix through matrices of the same size. By conjugating P with some involution, we show that if $P + J_n(I_n - P)$ is also positive (J_n is a matrix all of whose entries are $\frac{1}{n}$) then P is similar and strong shift equivalent over $\mathbb{U}_+$ to a doubly stochastic matrix through matrices of the same size. It is not true, however, that every positive stochastic matrix is strong shift equivalent over $\mathbb{R}_+$ to a doubly stochastic matrix of the same size: a counterexample was found by Johnson [7]. In Section 4, we prove that any positive stochastic matrix is strong shift equivalent over $\mathbb{U}_+$ to a positive doubly stochastic matrix, assuming $\mathbb{U}$ is a subring of $\mathbb{R}$ containing $\mathbb{Q}$. Consequently, the set of nonzero spectra of positive doubly stochastic matrices over a subfield $\mathbb{U}$ of $\mathbb{R}$ and the set of nonzero spectra of primitive stochastic matrices over $\mathbb{U}$ (see [3,13]) with positive trace coincide. We do not know whether the result can be extended to matrices which are primitive with zero trace.

The second aim of the paper is to give results and counterexamples for natural finiteness questions involving strong shift equivalence over $\mathbb{R}_+$. We prove that strong shift equivalence over $\mathbb{R}_+$ of irreducible stochastic matrices can be studied using generalized stochastic matrices in Section 5. We give a family of 2×2 matrices, pairwise strong shift equivalent over $\mathbb{R}_+$, such that there is no uniform bound on the minimum lag of a strong shift equivalence over $\mathbb{R}_+$ through matrices of bounded size between members of the family. In contrast, we prove that the collection of positive $n \times n$ matrices similar over $\mathbb{R}$ to any primitive $n \times n$ matrix of positive trace over $\mathbb{R}$ contains only finitely many SSE–$\mathbb{R}_+$ classes.