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Eigenvalues of discrete linear second-order periodic and antiperiodic eigenvalue problems with sign-changing weight

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ABSTRACT

By virtue of the eigenvalues of discrete linear second-order Neumann eigenvalue problems, we study the eigenvalues of discrete linear second-order periodic and antiperiodic eigenvalue problems with sign-changing weight. We find that these two problems have T real eigenvalues (including the multiplicity) respectively. Furthermore, the numbers of positive eigenvalues are equal to the numbers of positive elements in the weight function, and the numbers of negative eigenvalues are equal to the numbers of negative elements in the weight function. Furthermore, these eigenvalues, including the eigenvalues of Neumann problem, satisfy the order relation.

1. Introduction

Let $T > 2$ be an integer, $\mathbb{T} = \{1, 2, \cdots, T\}$. In this paper, we consider the eigenvalues of the following linear eigenvalue problem

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$$\Delta\big[p(t-1)\Delta u(t-1)\big] - q(t)u(t) + \lambda a(t)u(t) = 0, \quad t \in \mathbb{T}, \tag{1.1}$$

$$u(0) = u(T), \qquad u(1) = u(T+1), \tag{1.2}$$

where $p : \{0, 1, \cdots, T\} \to (0, \infty)$ satisfies $p(0) = p(T)$, $q : \mathbb{T} \to [0, \infty)$ and $a : \mathbb{T} \to \mathbb{R}$ satisfies the following condition:

(H0) a changes its sign on $\mathbb{T}$, i.e., there exists a proper subset, $\mathbb{T}_+$, of $\mathbb{T}$, such that

$$a(t) > 0 \quad \text{for } t \in \mathbb{T}_+, \qquad a(t) < 0 \quad \text{for } t \in \mathbb{T} \setminus \mathbb{T}_+.$$

Let n be the number of elements in $\mathbb{T}_+$. Then $T - n$ is the number of elements in $\mathbb{T}\setminus\mathbb{T}_+$. Further, we will consider the eigenvalues of (1.1) under the antiperiodic boundary condition, i.e., we will consider the eigenvalues of (1.1) under the boundary condition

$$u(0) + u(T) = 0, \qquad u(1) + u(T+1) = 0. \tag{1.3}$$

The study of the eigenvalue problem with sign-changing weight has lasted a long time. In 1914, Bôcher [1] studied the second-order differential equation eigenvalue problem

$$\frac{d}{dt}\big(ku'\big) + \big(\lambda m(t) - l\big)u(t) = 0, \quad t \in [0, 1], \tag{1.4}$$

$$\alpha u'(0) - \beta u(0) = 0, \qquad \gamma u'(1) + \delta u(1) = 0, \tag{1.5}$$

where m changes its sign on $[0, 1]$, $l \geq 0$ on $[0, 1]$. He obtained that (1.4), (1.5) has infinite real and simple eigenvalues $\lambda_k^{\pm}$, which satisfy

$$\cdots < \lambda_k^- < \cdots < \lambda_2^- < \lambda_1^- < 0 < \lambda_1^+ < \lambda_2^+ < \cdots < \lambda_k^+ < \cdots.$$

Hess and Kato [2] generalized this result to the second-order elliptic eigenvalue problem, and Anane et al. [3] generalized these results to the one-dimensional p-Laplacian eigenvalue problems. For the periodic eigenvalue problem with sign-changing weight, until 1997, Constantin [4] studied the following periodic eigenvalue problem

$$u''(t) = q(t)u(t) + \lambda m(t)u(t), \quad t \in [0, 1], \tag{1.6}$$

$$u(0) = u(1), \qquad u'(0) = u'(1), \tag{1.7}$$

where $m, q \in C[0, 1]$, $q(t) \geq 0$ and $q(t) \not\equiv 0$. He obtained that if m changes its sign, then the problem (1.6), (1.7) has infinite real eigenvalues, $\lambda_k^{\pm}$, such that

$$0 < \lambda_0^+ < \lambda_1^+ \leq \lambda_2^+ < \cdots \to +\infty$$

and

$$0 > \lambda_0^- > \lambda_1^- \geq \lambda_2^+ > \cdots \to -\infty.$$