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The set of space-filling curves: Topological and algebraic structure

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ABSTRACT

In this paper, a study of topological and algebraic properties of two families of functions from the unit interval I into the plane $\mathbb{R}^2$ is performed. The first family is the collection of all Peano curves, that is, of those continuous mappings onto the unit square. The second one is the bigger set of all space-filling curves, i.e. of those continuous functions $I \to \mathbb{R}^2$ whose images have the positive Jordan content. Emphasis is put on the size of these families, in both topological and algebraic senses, when endowed with natural structures.

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Spaceability
Algebrability

1. Introduction

In 1890 G. Peano [26] showed the existence of an astonishing mathematical object, namely, a curve filling the unit square. To be more precise, he constructed a *continuous surjective* mapping $I \to I^2$, where $I = [0,1]$ is the closed unit interval in the real line $\mathbb{R}$ and $I^2 = [0,1] \times [0,1]$.

Lebesgue [16,17,23] was probably the first to show an example of a function $f : \mathbb{R} \to \mathbb{R}$ that is *surjective* in a strong sense. Specifically, it satisfies $f(J) = \mathbb{R}$ for every nondegenerate interval J. Since then, many families of surjections $\mathbb{R} \to \mathbb{R}$, even in much stronger senses, have been presented (see [15,20,21]). Nevertheless, each of these functions is nowhere continuous. Of course, by using a bijection $\mathbb{R} \to I^2$ or $\mathbb{R} \to \mathbb{R}^2$, surjections $\mathbb{R} \to I^2$ or $\mathbb{R} \to \mathbb{R}^2$ (or even $I \to I^2$) can be constructed, but their continuity is far from being guaranteed.

Peano's result admits a topological extension, and in fact a topological characterization, which is given by the *Hahn–Mazurkiewicz theorem* (see e.g. [31, Theorem 31.5] or [19]): a Hausdorff topological space Y is a continuous image of the unit interval if and only if it is a compact, connected, locally connected, and second-countable space. Such a space Y is called a *Peano space*. Equivalently, by well-known metrization theorems, a Peano space is a compact, connected, locally connected metrizable topological space. Given two topological spaces X and Y, the set of continuous (continuous surjective, resp.) mappings $X \to Y$ will be denoted by $C(X,Y)$ ($CS(X,Y)$, resp.). Then the family of Peano curves is $\mathcal{P} := CS(I, I^2)$. If Y is a Peano space, we also denote $\mathcal{P}_Y := CS(I,Y)$, so that $\mathcal{P} = \mathcal{P}_{I^2}$.

There are several extensions of the notion of Peano curve on $\mathbb{R}^N$, with $N \geq 2$. Since the case $N = 2$ is illuminating enough, we will restrict ourselves to it. For instance, in [27], the next notion is given. By $c(A)$ it is denoted the Jordan content of a Jordan measurable set $A \subset \mathbb{R}^2$ (see Section 2 for definitions).

Definition 1.1. We say that a continuous function $\varphi : I \to \mathbb{R}^2$ is a *space-filling curve* provided that $\varphi(I)$ is Jordan measurable and $c(\varphi(I)) > 0$.

We can relax this condition by defining a λ-*space-filling curve* – where λ denotes Lebesgue measure on $\mathbb{R}^2$ – as a continuous function $f : I \to \mathbb{R}^2$ with $\lambda(f(I)) > 0$. This is not equivalent to the former definition; as a matter of fact, Osgood [25,27] constructed in 1903 a Jordan curve, that is, a continuous *injective* function $\psi : I \to \mathbb{R}^2$, such that $\lambda(\psi(I)) > 0$; here $\psi(I)$ cannot be Jordan measurable. Other related notions can be found in [24] and [30]. The symbol $\mathcal{SF}$ will stand for the set of all space-filling curves in the sense of Definition 1.1.