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Euclidean distance degrees of real algebraic groups

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A B S T R A C T

We study the problem of finding, in a real algebraic matrix group, the matrix nearest to a given data matrix. We do so from the algebro-geometric perspective of Euclidean distance degrees. We recover several classical results, and among the new results that we prove is a formula for the Euclidean distance degree of special linear groups.

1. The distance to a matrix group

In many applications across mathematics and engineering, one encounters the problem of finding the point on a real algebraic variety that minimises the distance to a given data point outside the variety, and of counting the critical points on the variety of the distance function to the data point. In this paper we discuss the special case of these problems where the variety is a real algebraic group realised as a matrix group. We

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review known results for orthogonal and unitary groups, prove a general upper bound for compact tori, and discuss the problem of finding the nearest determinant-one matrix and the nearest symplectic matrix.

Let V be an n-dimensional real vector space equipped with a positive definite inner product $(.|.)$, and write $\mathrm{End}(V)$ for the space of linear maps $V \to V$. The inner product gives rise to a linear map $\mathrm{End}(V) \to \mathrm{End}(V)$, $a \mapsto a^t$ called transposition and determined by the property that $(av|w) = (v|a^t w)$ for all $v, w \in V$, and also to a positive definite inner product $\langle .,. \rangle$ on $\mathrm{End}(V)$ defined by $\langle a, b \rangle := \mathrm{tr}(a^t b)$. This inner product enjoys properties such as $\langle a, bc \rangle = \langle b^t a, c \rangle$. The associated norm $\|.\|$ on $\mathrm{End}(V)$ is called the Frobenius norm. If we choose an orthonormal basis of V and denote the entries of the matrix of $a \in \mathrm{End}(V)$ relative to this basis by a_{ij}, then $\|a\|^2 = \sum_{ij} a_{ij}^2$. We will use the words matrix and linear maps interchangeably, but we work without choosing coordinates because it allows for a more elegant statement of some of the results. For $a, b \in \mathrm{End}(V)$ and $v, w \in V$ we write $a \perp b$ for $\langle a, b \rangle = 0$, and $v \perp w$ for $(v|w) = 0$.

Let G be a Zariski-closed subgroup of the real algebraic group $\mathrm{GL}(V) \subseteq \mathrm{End}(V)$ of invertible linear maps. In other words, G is a subgroup of $\mathrm{GL}(V)$ characterised by polynomial equations in the matrix entries. Then G is a real algebraic group and in particular a smooth manifold. The problem motivating this note is the following.

Problem 1.1. Given a general $u \in \mathrm{End}(V)$, determine $x \in G$ that minimises the squared-distance function $d_u(x) := \|u - x\|^2$.

Here, and in the rest of this note, *general* means that whenever convenient, we may assume that u lies outside some proper, Zariski-closed subset of $\mathrm{End}(V)$. Instances of this problem appear naturally in applications. For instance, the *nearest orthogonal matrix* plays a role in computer vision [6], and we revisit its solution in Section 3. More or less equivalent to this is the solution to the orthogonal Procrustes problem [11]. For these and other matrix nearness problems we refer to [5,8]. More recent applications include structured low-rank approximation, for which algebraic techniques are developed in [10].

The bulk of this note is devoted to *counting* the number of critical points on G of the function d_u, in the general framework of the *Euclidean distance degree* (ED degree) [2]. In Section 2 we specialise this framework to matrix groups. In Section 3 we discuss matrix groups preserving the inner product. In particular, we derive a conjecturally sharp upper bound on the ED degree of a compact torus preserving the inner product, revisit the classical cases of orthogonal and unitary groups, and express the ED degree as the algebraic degree of a certain matrix multiplication map. Then in Section 4 we discuss two classes of groups not preserving the inner product: the special linear groups, consisting of all determinant-one matrices, and the symplectic groups. For the former we determine the ED degree explicitly. We conclude the note with a conjecture for the latter.