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Infinite random matrix theory, tridiagonal bordered Toeplitz matrices, and the moment problem

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ABSTRACT

The four major asymptotic level density laws of random matrix theory may all be showcased through their Jacobi parameter representation as having a bordered Toeplitz form. We compare and contrast these laws, completing and exploring their representations in one place. Inspired by the bordered Toeplitz form, we propose an algorithm for the finite moment problem by proposing a solution whose density has a bordered Toeplitz form.

1. Introduction

Consider the “big” laws for asymptotic level densities for various random matrices:

Wigner semicircle law	[19]
Marchenko–Pastur law	[13]
Kesten–McKay law	[10,14]
Wachter law	[18]

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Table 1
Random matrix laws in raw form. The Kesten–McKay and Wachter laws are related by the linear transform $(2x_{\text{Wachter}} - 1)v = x_{\text{Kesten–McKay}}$ and $a = b = v/2$.

Measure	Support	Parameters
Wigner semicircle $d\mu_{WS} = \frac{\sqrt{4-x^2}}{2\pi}dx$	$I_{WS} = [\pm 2]$	N/A
Marchenko–Pastur $d\mu_{MP} = \frac{\sqrt{(\lambda_+ - x)(x-\lambda_-)}}{2\pi x}dx$	$I_{MP} = [\lambda_-, \lambda_+]$	$\lambda_\pm = (1 \pm \sqrt{\lambda})^2$, $\lambda \geq 1$
Kesten–McKay $d\mu_{KM} = \frac{v\sqrt{4(v-1)-x^2}}{2\pi(v^2-x^2)}dx$	$I_M = [\pm 2\sqrt{v-1}]$	$v \geq 2$
Wachter $d\mu_W = \frac{(a+b)\sqrt{(\mu_+ - x)(x-\mu_-)}}{2\pi x(1-x)}dx$	$I_W = [\mu_-, \mu_+]$	$\mu_\pm = (\frac{\sqrt{b} \pm \sqrt{a(a+b-1)}}{a+b})^2$, $a, b \geq 1$

In raw form, these laws (Table 1) appear as somewhat complicated expressions involving square roots. This paper highlights a unifying principle that underlies these four laws, namely the laws may be encoded as Jacobi symmetric tridiagonal matrices that are Toeplitz with a length 1 boundary.

This suggests that some of the nice properties of the big laws are connected to this property, and further suggests the importance of the larger family of laws encoded as Toeplitz with length k boundary, known as "nearly Toeplitz" matrices. This motivates the two parts of this paper:

(1) We tabulate in one place key properties of the four laws, not all of which can be found in the literature. These sections are expository, with the exception of the as-of-yet unpublished Wachter moments, and the Kesten–McKay and Wachter law Jacobi parameters and free cumulants.
(2) We describe a new algorithm to exploit the Toeplitz-with-length-k boundary structure. In particular, we show how practical it is to approximate distributions with incomplete information using distributions having nearly-Toeplitz encodings.

Studies of nearly Toeplitz matrices in random matrix theory have been pioneered by Anshelevich [1,2].

Historically, the Wigner semicircle law is the most famous. The weight function is classical, and corresponds to Chebychev polynomials of the second kind. It is the equilibrium measure [3] for Hermite polynomials and the asymptotic distribution for Gaussian or Hermite ensembles (GOE, GUE, GSE, etc.). None of the other weight functions are classical, but they are all equilibrium measures for classical polynomials. The second most famous law is the Marchenko–Pastur law. It is the equilibrium measure for Laguerre polynomials and is the asymptotic distribution for Wishart matrices or Laguerre ensembles. The Kesten–McKay law, described in [9], is the equilibrium measure for Gegenbauer polynomials. It is not commonly included among the Wigner semicircle, Marchenko–Pastur, and Wachter laws, but we believe that it merits inclusion on account of its place in the