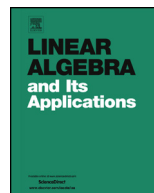




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# Jordan derivations of prime rings with characteristic two

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## ABSTRACT

Let  $R$  be a prime ring with center  $Z(R)$  and with extended centroid  $C$ . We give a complete characterization of Jordan derivations of  $R$  when  $\text{char } R = 2$  and  $\dim_C RC = 4$ : An additive map  $\delta: R \rightarrow RC$  is a Jordan derivation if and only if there exist a derivation  $d: R \rightarrow RC$  and an additive map  $\mu: R \rightarrow C$  such that  $\delta = d + \mu$  and  $\mu(x^2) = 0$  for all  $x \in R$ . As consequences, it is proved among other things: Any  $Z(R)$ -linear Jordan derivation of  $R$  is a derivation if  $\dim_C RC < \infty$ . Moreover, if  $C$  is either a finite field or an algebraically closed field, where  $\text{char } C = 2$  and  $n \geq 2$ , then every Jordan derivation of  $M_n(C)$  is a derivation.

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## 1. Introduction

Throughout the paper,  $R$  always denotes a prime ring; that is, for  $a, b \in R$ , if  $aRb = 0$  then either  $a = 0$  or  $b = 0$ . A map  $d: R \rightarrow R$  is called a derivation if  $d(x+y) = d(x) + d(y)$  and  $d(xy) = d(x)y + xd(y)$  for all  $x, y \in R$ . Clearly, given  $b \in R$ , the map  $x \mapsto [b, x]$  for  $x \in R$  is a derivation, where  $[b, x] := bx - xb$ . It is called the inner derivation of  $R$  defined by the element  $b$ . By a Jordan derivation we mean an additive map  $\delta: R \rightarrow R$

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satisfying  $\delta(x^2) = \delta(x)x + x\delta(x)$  for all  $x \in R$ . Every derivation is a Jordan derivation. However, a Jordan derivation is not in general a derivation. In 1957 Herstein proved that if  $\text{char } R \neq 2$ , then any Jordan derivation of  $R$  is a derivation (see [10, Theorem 3.1]). We refer the reader to [9] and [3] for the 2-torsion free semiprime case. For the case  $\text{char } R = 2$ , Herstein proved that if  $R$  is not a commutative integral domain, then a Jordan derivation  $\delta$  of  $R$  satisfying

$$\delta(xy) = \delta(x)y + x\delta(y) \quad \text{for all } x, y \in R \tag{†}$$

is a derivation (see [10, Theorem 4.1]). See also [4] for further results concerning Jordan derivations satisfying (†). For our purpose, we give a slight generalization concerning (Jordan) derivations. Let  $R \subseteq S$  be rings. An additive map  $\delta: R \rightarrow S$  is called a derivation (resp. Jordan derivation) if  $\delta(xy) = \delta(x)y + x\delta(y)$  (resp.  $\delta(x^2) = \delta(x)x + x\delta(x)$ ) for all  $x, y \in R$ . Let  $Q_{ml}(R)$  denote the maximal left ring of quotients of  $R$  and let  $C$  denote the center of  $Q_{ml}(R)$ . It is known that  $Q_{ml}(R)$  is also a prime ring and  $C$  is a field, which is called the extended centroid of  $R$ . We refer the reader to the book [2] for details. For our purpose, we restate Herstein’s theorem [10, Theorem 3.1] as follows (it is still true with the same proof):

**Theorem 1.1** (Herstein). *Let  $R$  be a prime ring of characteristic different from 2. Then every Jordan derivation  $\delta: R \rightarrow Q_{ml}(R)$  is a derivation.*

By [5, Corollary 6.9], we have the following.

**Theorem 1.2.** *Let  $R$  be a prime ring. Suppose that  $\delta: R \rightarrow Q_{ml}(R)$  is a Jordan derivation. Then there exist a derivation  $d: R \rightarrow Q_{ml}(R)$  and an additive map  $\mu: R \rightarrow C$  such that  $\delta = d + \mu$  and  $\mu(x^2) = 0$  for all  $x \in R$  except when  $\dim_C RC = 4$  and  $\text{char } R = 2$ .*

**Proof.** By Theorem 1.1, we may assume that  $\text{char } R = 2$ . Suppose that  $R$  is commutative. Then  $\delta(x^2) = x\delta(x) + \delta(x)x = 2x\delta(x) = 0$  for  $x \in R$ . Set  $d := 0$  and  $\mu := \delta$ , as asserted. Hence, we may assume that  $\text{char } R = 2$  and  $\dim_C RC > 4$ . Linearizing  $\delta(x^2) = x\delta(x) + \delta(x)x$  and using the fact that  $\text{char } R = 2$ , we see that  $\delta$  is a Lie derivation. That is,

$$\delta([x, y]) = [\delta(x), y] + [x, \delta(y)]$$

for all  $x, y \in R$ . Note that [5, Corollary 6.9] keeps true when “ $\delta: R \rightarrow R$ ” is replaced by “ $\delta: R \rightarrow Q_{ml}(R)$ ”. Thus, there exist a derivation  $d: R \rightarrow Q_{ml}(R)$  and an additive map  $\mu: R \rightarrow C$  such that  $\delta = d + \mu$  and  $\mu([x, y]) = 0$  for all  $x, y \in R$ . Let  $x \in R$ . Then

$$\delta(x^2) = [\delta(x), x] = [d(x) + \mu(x), x] = [d(x), x] = d(x^2).$$

On the other hand,  $\delta(x^2) = d(x^2) + \mu(x^2)$ . Thus,  $\mu(x^2) = 0$  for all  $x \in R$ .  $\square$

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