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Jordan derivations of prime rings with characteristic two

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ABSTRACT

Let R be a prime ring with center $Z(R)$ and with extended centroid C. We give a complete characterization of Jordan derivations of R when $\operatorname{char} R = 2$ and $\dim_C RC = 4$: An additive map $\delta: R \to RC$ is a Jordan derivation if and only if there exist a derivation $d: R \to RC$ and an additive map $\mu: R \to C$ such that $\delta = d + \mu$ and $\mu(x^2) = 0$ for all $x \in R$. As consequences, it is proved among other things: Any $Z(R)$-linear Jordan derivation of R is a derivation if $\dim_C RC < \infty$. Moreover, if C is either a finite field or an algebraically closed field, where $\operatorname{char} C = 2$ and $n \geq 2$, then every Jordan derivation of $\mathrm{M}_n(C)$ is a derivation.

1. Introduction

Throughout the paper, R always denotes a prime ring; that is, for $a, b \in R$, if $aRb = 0$ then either $a = 0$ or $b = 0$. A map $d: R \to R$ is called a derivation if $d(x+y) = d(x)+d(y)$ and $d(xy) = d(x)y + xd(y)$ for all $x, y \in R$. Clearly, given $b \in R$, the map $x \mapsto [b, x]$ for $x \in R$ is a derivation, where $[b, x] := bx - xb$. It is called the inner derivation of R defined by the element b. By a Jordan derivation we mean an additive map $\delta: R \to R$

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satisfying $\delta(x^2) = \delta(x)x + x\delta(x)$ for all $x \in R$. Every derivation is a Jordan derivation. However, a Jordan derivation is not in general a derivation. In 1957 Herstein proved that if $\operatorname{char} R \neq 2$, then any Jordan derivation of R is a derivation (see [10, Theorem 3.1]). We refer the reader to [9] and [3] for the 2-torsion free semiprime case. For the case $\operatorname{char} R = 2$, Herstein proved that if R is not a commutative integral domain, then a Jordan derivation δ of R satisfying

$$\delta(xyx) = \delta(x)yx + x\delta(y)x + xy\delta(x) \quad \text{for all } x, y \in R \tag{†}$$

is a derivation (see [10, Theorem 4.1]). See also [4] for further results concerning Jordan derivations satisfying (†). For our purpose, we give a slight generalization concerning (Jordan) derivations. Let $R \subseteq S$ be rings. An additive map $\delta: R \to S$ is called a derivation (resp. Jordan derivation) if $\delta(xy) = \delta(x)y + x\delta(y)$ (resp. $\delta(x^2) = \delta(x)x + x\delta(x)$) for all $x, y \in R$. Let $Q_{ml}(R)$ denote the maximal left ring of quotients of R and let C denote the center of $Q_{ml}(R)$. It is known that $Q_{ml}(R)$ is also a prime ring and C is a field, which is called the extended centroid of R. We refer the reader to the book [2] for details. For our purpose, we restate Herstein's theorem [10, Theorem 3.1] as follows (it is still true with the same proof):

Theorem 1.1 *(Herstein). Let R be a prime ring of characteristic different from 2. Then every Jordan derivation $\delta: R \to Q_{ml}(R)$ is a derivation.*

By [5, Corollary 6.9], we have the following.

Theorem 1.2. *Let R be a prime ring. Suppose that $\delta: R \to Q_{ml}(R)$ is a Jordan derivation. Then there exist a derivation $d: R \to Q_{ml}(R)$ and an additive map $\mu: R \to C$ such that $\delta = d + \mu$ and $\mu(x^2) = 0$ for all $x \in R$ except when $\dim_C RC = 4$ and $\operatorname{char} R = 2$.*

Proof. By Theorem 1.1, we may assume that $\operatorname{char} R = 2$. Suppose that R is commutative. Then $\delta(x^2) = x\delta(x) + \delta(x)x = 2x\delta(x) = 0$ for $x \in R$. Set $d := 0$ and $\mu := \delta$, as asserted. Hence, we may assume that $\operatorname{char} R = 2$ and $\dim_C RC > 4$. Linearizing $\delta(x^2) = x\delta(x) + \delta(x)x$ and using the fact that $\operatorname{char} R = 2$, we see that δ is a Lie derivation. That is,

$$\delta([x, y]) = \big[\delta(x), y\big] + \big[x, \delta(y)\big]$$

for all $x, y \in R$. Note that [5, Corollary 6.9] keeps true when "$\delta: R \to R$" is replaced by "$\delta: R \to Q_{ml}(R)$". Thus, there exist a derivation $d: R \to Q_{ml}(R)$ and an additive map $\mu: R \to C$ such that $\delta = d + \mu$ and $\mu([x, y]) = 0$ for all $x, y \in R$. Let $x \in R$. Then

$$\delta\big(x^2\big) = \big[\delta(x), x\big] = \big[d(x) + \mu(x), x\big] = \big[d(x), x\big] = d\big(x^2\big).$$

On the other hand, $\delta(x^2) = d(x^2) + \mu(x^2)$. Thus, $\mu(x^2) = 0$ for all $x \in R$. □