Linear Algebra and its Applications 462 (2014) 16–38

Contents lists available at ScienceDirect

Linear Algebra and its Applications

www.elsevier.com/locate/laa

Affine transformations of a sharp tridiagonal pair

Bo Hou, Longmei Yang, Suogang Gao *

College of Mathematics and Information Science, Hebei Normal University, Shijiazhuang, 050024, PR China

ARTICLE INFO

Article history:
Received 13 June 2014
Accepted 16 August 2014
Available online 28 August 2014
Submitted by R. Brualdi

MSC:
primary 05E30
secondary 15A21, 33C45, 33D45

Keywords:
Tridiagonal pair
Tridiagonal system
Affine transformation
Parameter array
Sharp

ABSTRACT

Let $\mathbb{K}$ denote a field and let V denote a vector space over $\mathbb{K}$ with finite positive dimension. By a *tridiagonal pair*, we mean an ordered pair A, A^* of $\mathbb{K}$-linear transformations from V to V that satisfy the following conditions: (i) each of A, A^* is diagonalizable; (ii) there exists an ordering $\{V_i\}_{i=0}^d$ of the eigenspaces of A such that $A^*V_i \subseteq V_{i-1} + V_i + V_{i+1}$ $(0 \leq i \leq d)$, where $V_{-1} = 0$, $V_{d+1} = 0$; (iii) there exists an ordering $\{V_i^*\}_{i=0}^\delta$ of the eigenspaces of A^* such that $AV_i^* \subseteq V_{i-1}^* + V_i^* + V_{i+1}^*$ $(0 \leq i \leq \delta)$, where $V_{-1}^* = 0$, $V_{\delta+1}^* = 0$; (iv) there is no subspace W of V such that $AW \subseteq W$, $A^*W \subseteq W$, $W \neq 0$, $W \neq V$. It is known that $\eta A + \mu I$, $\eta^* A^* + \mu^* I$ is also a tridiagonal pair on V, where η, μ, η^*, μ^* are scalars in $\mathbb{K}$ with η, η^* nonzero. In this paper we give the necessary and sufficient conditions for these tridiagonal pairs to be isomorphic to A, A^* or A^*, A. We do this under a mild assumption, called the sharp condition.

1. Introduction

Throughout the paper $\mathbb{K}$ denotes a field and V denotes a vector space over $\mathbb{K}$ with finite positive dimension.

* Corresponding author.
E-mail addresses: sggaomail@163.com, sggao@hebtu.edu.cn (S. Gao).

http://dx.doi.org/10.1016/j.laa.2014.08.012
0024-3795/© 2014 Published by Elsevier Inc.

We begin by recalling the notion of a *tridiagonal pair*. We will use the following terms. Let $\mathrm{End}(V)$ denote the $\mathbb{K}$-algebra consisting of all $\mathbb{K}$-linear transformations from V to V. For $A \in \mathrm{End}(V)$ and for subspace $W \subseteq V$, we call W an *eigenspace* of A whenever $W \neq 0$ and there exists $\theta \in \mathbb{K}$ such that $W = \{v \in V \mid Av = \theta v\}$; in this case θ is the eigenvalue of A associated with W. We say A is *diagonalizable* whenever V is spanned by the eigenspaces of A.

Definition 1.1. (See [1, Definition 1.1].) By a *tridiagonal pair* on V, we mean an ordered pair $A, A^* \in \mathrm{End}(V)$ that satisfy (i)–(iv) below:

(i) Each of A, A^* is diagonalizable.
(ii) There exists an ordering $\{V_i\}_{i=0}^{d}$ of the eigenspaces of A such that

$$A^* V_i \subseteq V_{i-1} + V_i + V_{i+1} \quad (0 \leq i \leq d), \tag{1}$$

where $V_{-1} = 0$, $V_{d+1} = 0$.
(iii) There exists an ordering $\{V_i^*\}_{i=0}^{\delta}$ of the eigenspaces of A^* such that

$$A V_i^* \subseteq V_{i-1}^* + V_i^* + V_{i+1}^* \quad (0 \leq i \leq \delta), \tag{2}$$

where $V_{-1}^* = 0$, $V_{\delta+1}^* = 0$.
(iv) There is no subspace W of V such that $AW \subseteq W$, $A^*W \subseteq W$, $W \neq 0$, $W \neq V$.

We say the pair A, A^* is *over* $\mathbb{K}$.

Let A, A^* denote a tridiagonal pair on V. Then A^*, A is also a tridiagonal pair on V, we call A^*, A the *dual* of A, A^*.

Let A, A^* denote a tridiagonal pair on V. By [1, Lemma 4.5], the integers d and δ from Definition 1.1 are equal; we call this common value the *diameter* of A, A^*. An ordering of the eigenspaces of A (resp. A^*) is said to be *standard* whenever it satisfies (1) (resp. (2)). By [1, Corollary 5.7], for $0 \leq i \leq d$ the spaces V_i, V_i^* have the same dimension; we denote this common dimension by ρ_i. We call the sequence $\{\rho_i\}_{i=0}^{d}$ the *shape* of A, A^*. By [1, Corollaries 5.7, 6.6], $\{\rho_i\}_{i=0}^{d}$ is symmetric and unimodal; that is $\rho_i = \rho_{d-i}$ for $0 \leq i \leq d$ and $\rho_{i-1} \leq \rho_i$ for $0 \leq i \leq \frac{d}{2}$. The pair A, A^* is said to be *sharp* whenever $\rho_0 = 1$ [4, Definition 1.5]. By a *Leonard pair*, we mean a tridiagonal pair such that $\rho_i = 1$ for $0 \leq i \leq d$ [5, Definition 1.1].

In the survey paper [7], P. Terwilliger proposed the following open problem.

Problem 1.2. [7, Problem 36.1] Let A, A^* denote a tridiagonal pair on V. Let η, μ, η^*, μ^* denote scalars in $\mathbb{K}$ with η, η^* nonzero. Note that the pair $\eta A + \mu I, \eta^* A^* + \mu^* I$ is a tridiagonal pair on V. Find necessary and sufficient conditions for this tridiagonal pair to be isomorphic to the tridiagonal pair A, A^*. Also, find necessary and sufficient conditions for this tridiagonal pair to be isomorphic to the tridiagonal pair A^*, A.