Linear Algebra and its Applications 462 (2014) 39–58

Contents lists available at ScienceDirect

Linear Algebra and its Applications

www.elsevier.com/locate/laa

The spectra of multiplicative attribute graphs

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A R T I C L E I N F O

Article history:
Received 22 February 2012
Accepted 29 July 2014
Available online 30 August 2014
Submitted by R. Brualdi

MSC:
primary 05C80
secondary 05C82, 60B20

Keywords:
Stochastic Kronecker graphs
Multiplicative attribute graphs
Normalized Laplacian
Graph spectra

A B S T R A C T

A multiplicative attribute graph is a random graph in which vertices are represented by random words of length t in a finite alphabet Γ, and the probability of adjacency is a symmetric function $\Gamma^t \times \Gamma^t \rightarrow [0, 1]$. These graphs are a generalization of stochastic Kronecker graphs, and both classes have been shown to exhibit several useful real world properties. We establish asymptotic bounds on the spectra of the adjacency matrix and the normalized Laplacian matrix for these two families of graphs under certain mild conditions. As an application we examine various properties of the stochastic Kronecker graph and the multiplicative attribute graph, including the diameter, clustering coefficient, chromatic number, and bounds on low-congestion routing.

1. Introduction

Over the last few decades, spurred by the attempts to understand and model modern complex networks, there has been an extensive amount of literature devoted to studying models for random graphs that differ significantly from the standard Erdős–Rényi

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http://dx.doi.org/10.1016/j.laa.2014.07.047
0024-3795/

random graph and the d-regular random graph (see for instance [5,9,11,18,30]). One trend that has begun to emerge among these random graph models is the use of mathematical primitives to create models for complex networks that exhibit complicated behavior while still being analytically tractable. For example, inhomogeneous random graphs [3,4] and random dot product graphs [26,32,41,44,45], both build graphs over an inner product space and use the inner product to govern the edge connectivity, while stochastic Kronecker graphs [28,29] and multiplicative attribute graphs [25] use the Kronecker product of matrices to control the edge probabilities. In this work we focus on the spectral properties of the latter two random graph models, showing that these properties can be derived in a natural way using the Kronecker product.

More formally, the Kronecker product of matrices $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{p\times q}$ is a matrix $A \otimes B = C \in \mathbb{R}^{mp\times nq}$ where $C_{i,j} = A_{\lceil \frac{i}{m} \rceil, \lceil \frac{j}{n} \rceil} B_{i \bmod p, j \bmod q}$ and $x \bmod p \in [p] = \{1, 2, \ldots, p\}$. That is,

$$A \otimes B = C = \begin{bmatrix} A_{1,1}B & A_{1,2}B & \cdots & A_{1,n}B \\ A_{2,1}B & A_{2,2}B & \cdots & A_{2,n}B \\ \cdots & \cdots & \ddots & \cdots \\ A_{m,1}B & A_{m,2}B & \cdots & A_{m,n}B \end{bmatrix}.$$

A stochastic Kronecker graph is formed by taking a symmetric $k \times k$ matrix P with entries in the interval $[0, 1]$ and a positive integer t, and forming the t-fold Kronecker product, denoted $P^{\otimes t}$. Each edge $\{i, j\}$ is then present independently with probability $P^{\otimes t}_{i,j} = P^{\otimes t}_{j,i}$. We will say that such a graph is a t^{th}-order stochastic Kronecker graph with generating matrix P. Recently, the stochastic Kronecker graph has been advanced as a model for the internet and other complex networks [29] especially in the case where the generating matrix is $\left[\begin{smallmatrix} \alpha & \beta \\ \beta & \gamma \end{smallmatrix}\right]$ and $\alpha \geq \beta \geq \gamma$. Mahdian and Xu have recently analyzed the connectivity, diameter, and emergence of the giant component in this context [31] while the first author and Horn analyzed the emergence of the giant component of a general 2×2 generating matrix [37]. The multiplicative attribute graph is a natural generalization of stochastic Kronecker graphs to allow multiple copies of each vertex before determining the random edges. In order to make this precise, we equip the $k \times k$ generating matrix for the stochastic Kronecker graph with an alphabet Γ of size k, and define a function $w : V \to \Gamma^t$. Then any two vertices $u, v \in V$ are connected independently with probability $P^{\otimes t}_{w(u),w(v)}$. We note here that u may be equal to v, so that we allow self-edges with the appropriate probability. If the function w is a bijection, then this is a t^{th}-order stochastic Kronecker graph with generating matrix P, while if w is not a bijection we say that the resulting graph is a t^{th}-order multiplicative attribute graph with generating matrix P.

We provide here asymptotic bounds on the spectra of both the stochastic Kronecker graph and a generalization due to Kim and Leskovec [25] known as the multiplicative attribute graph. Moreover, we show some applications of these bounds to graph properties related to spectra. Although there are several natural spectra of graphs to consider,