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The end-parameters of a Leonard pair

Kazumasa Nomura

Tokyo Medical and Dental University, Kohnodai, Ichikawa, 272-0827 Japan

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A B S T R A C T

Fix an algebraically closed field $\mathbb{F}$ and an integer $d \geq 3$. Let V be a vector space over $\mathbb{F}$ with dimension $d+1$. A Leonard pair on V is a pair of diagonalizable linear transformations $A : V \to V$ and $A^* : V \to V$, each acting in an irreducible tridiagonal fashion on an eigenbasis for the other one. There is an object related to a Leonard pair called a Leonard system. It is known that a Leonard system is determined up to isomorphism by a sequence of scalars $(\{\theta_i\}_{i=0}^d, \{\theta^*_i\}_{i=0}^d, \{\varphi_i\}_{i=1}^d, \{\phi_i\}_{i=1}^d)$, called its parameter array. The scalars $\{\theta_i\}_{i=0}^d$ (resp. $\{\theta^*_i\}_{i=0}^d$) are mutually distinct, and the expressions $(\theta_{i-2} - \theta_{i+1})/(\theta_{i-1} - \theta_i)$, $(\theta^*_{i-2} - \theta^*_{i+1})/(\theta^*_{i-1} - \theta^*_i)$ are equal and independent of i for $2 \leq i \leq d-1$. Write this common value as $\beta+1$. In the present paper, we consider the "end-parameters" $\theta_0, \theta_d, \theta^*_0, \theta^*_d, \varphi_1, \varphi_d, \phi_1, \phi_d$ of the parameter array. We show that a Leonard system is determined up to isomorphism by the end-parameters and β. We display a relation between the end-parameters and β. Using this relation, we show that there are up to isomorphism at most $\lfloor (d-1)/2 \rfloor$ Leonard systems that have specified end-parameters. The upper bound $\lfloor (d-1)/2 \rfloor$ is best possible.

1. Introduction

Throughout the paper $\mathbb{F}$ denotes an algebraically closed field.

E-mail address: knomura@pop11.odn.ne.jp.

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We begin by recalling the notion of a Leonard pair. We use the following terms. A square matrix is said to be *tridiagonal* whenever each nonzero entry lies on either the diagonal, the subdiagonal, or the superdiagonal. A tridiagonal matrix is said to be *irreducible* whenever each entry on the subdiagonal is nonzero and each entry on the superdiagonal is nonzero.

Definition 1.1. (See [5, Definition 1.1].) Let V be a vector space over $\mathbb{F}$ with finite positive dimension. By a *Leonard pair on* V we mean an ordered pair of linear transformations $A : V \to V$ and $A^* : V \to V$ that satisfy (i) and (ii) below:

(i) There exists a basis for V with respect to which the matrix representing A is irreducible tridiagonal and the matrix representing A^* is diagonal.
(ii) There exists a basis for V with respect to which the matrix representing A^* is irreducible tridiagonal and the matrix representing A is diagonal.

Note 1.2. According to a common notational convention, A^* denotes the conjugate transpose of A. We are not using this convention. In a Leonard pair A, A^* the matrices A and A^* are arbitrary subject to the conditions (i) and (ii) above.

We refer the reader to [3,5–8] for background on Leonard pairs.

For the rest of this section, fix an integer $d \geq 0$ and a vector space V over $\mathbb{F}$ with dimension $d+1$. Consider a Leonard pair A, A^* on V. By [5, Lemma 1.3] each of A, A^* has mutually distinct $d+1$ eigenvalues. Let $\{\theta_i\}_{i=0}^d$ be an ordering of the eigenvalues of A, and let $\{V_i\}_{i=0}^d$ be the corresponding eigenspaces. For $0 \leq i \leq d$ define $E_i : V \to V$ such that $(E_i - I)V_i = 0$ and $E_i V_j = 0$ for $j \neq i$ $(0 \leq j \leq d)$. Here I denotes the identity. We call E_i the *primitive idempotent* of A associated with θ_i. The primitive idempotent E_i^* of A^* associated with θ_i^* is similarly defined. For $0 \leq i \leq d$ pick a nonzero $v_i \in V_i$. Note that $\{v_i\}_{i=0}^d$ is a basis for V. We say the ordering $\{E_i\}_{i=0}^d$ is *standard* whenever the basis $\{v_i\}_{i=0}^d$ satisfies Definition 1.1(ii). A standard ordering of the primitive idempotents of A^* is similarly defined. For a standard ordering $\{E_i\}_{i=0}^d$, the ordering $\{E_{d-i}\}_{i=0}^d$ is also standard and no further ordering is standard. A similar result applies to a standard ordering of the primitive idempotents of A^*.

Definition 1.3. (See [5, Definition 1.4].) By a *Leonard system* on V we mean a sequence

$$\Phi = \big(A, \{E_i\}_{i=0}^d, A^*, \{E_i^*\}_{i=0}^d\big), \tag{1}$$

where A, A^* is a Leonard pair on V, and $\{E_i\}_{i=0}^d$ (resp. $\{E_i^*\}_{i=0}^d$) is a standard ordering of the primitive idempotents of A (resp. A^*). We say Φ *is over* $\mathbb{F}$. We call d the *diameter* of Φ.

We recall the notion of an isomorphism of Leonard systems. Consider a Leonard system (1) on V and a Leonard system $\Phi' = (A', \{E_i'\}_{i=0}^d, A^{*\prime}, \{E_i^{*\prime}\}_{i=0}^d)$ on a vector space V' with dimension $d+1$. By an *isomorphism of Leonard systems from* Φ *to* Φ' we