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Essential spectra of linear relations

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ABSTRACT

Five essential spectra of linear relations are defined in terms of semi-Fredholm properties and the index. Basic properties of these sets are established and the perturbation theory for semi-Fredholm relations is then applied to verify a generalisation of Weyl's theorem for single-valued operators. We conclude with a Möbius transform spectral mapping theorem.

1. Introduction

While the study of the spectrum of bounded linear operators generalises the theory of eigenvalues of matrices, the essential spectra of linear operators characterise the non-invertibility of operators $\lambda - T$. The latter have been considered in terms of two key related directions of investigation, namely the study of the ascent and descent (as well

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as the nullity and defect) of $\lambda - T$ and in terms of semi-Fredholm properties of $\lambda - T$. Today there are several related definitions of essential spectra and comprehensive reviews may be found in [9,11,15–17,19,22,26]. In [13], the refinements of the spectrum in terms of ascent and descent were investigated in terms of states of operators, using the terminology of [23] (see [10] and also [6] for the states of linear relations). On the other hand, the perturbation theory of semi-Fredholm operators provides a more general context for the early observations of H. Weyl, who showed that limit points of the spectrum (i.e. all points of the spectrum, except isolated eigenvalues of finite multiplicity) of a bounded symmetric transformation on a Hilbert space are invariant under perturbation by compact symmetric operators ([24], cf. Riesz and Sz.-Nagy [20]).

In this paper we apply the theory of Fredholm relations to show that the theory for essential spectra of linear operators can be extended naturally to linear relations. In particular, we extend preliminary results of Cross [6] by generalising the definitions given in Edmunds and Evans [7] for single-valued operators.

We commence with a recollection of some preliminary properties required in the sequel.

2. Semi-Fredholm linear relations

We first clarify some notation and terminology. Let X and Y be normed linear spaces, and let $B(X,Y)$ and $L(X,Y)$ denote the classes of bounded and unbounded linear operators, respectively, from X into Y. A **multivalued linear operator** $T : X \to Y$ is a set-valued map such that its graph $G(T) = \{(x,y) \in X \times Y \mid y \in Tx\}$ is a linear subspace of $X \times Y$. We use the term **linear relation** or simply relation, to refer to such a multivalued linear operator denoted by $T \in LR(X,Y)$ (cf. Arens [2] and Lee and Nashed [14]). A relation $T \in LR(X,Y)$ is said to be **closed** if its graph $G(T)$ is a closed subspace. The **closure** of a linear relation T, denoted by $\overline{T}$ is defined in terms of its corresponding graph: $G(\overline{T}) := \overline{G(T)} \subset X \times Y$.

The **conjugate** T' (cf. [6], III.1.1) of a linear relation $T \in LR(X,Y)$ is defined by

$$G(T') := G(-T^{-1})^{\perp} \subset Y' \times X'$$

where $[(y,x),(y',x')] := [x,x'] + [y,y'] = x'x + y'y$. For $(y',x') \in G(T')$ we have $y'y = x'x$ whenever $x \in D(T)$.

Let Q_T, or simply Q, when there is no ambiguity about the relation T, denote the natural quotient map $Q^{Y}_{\overline{T(0)}} : Y \to Y/\overline{T(0)}$ with kernel $\overline{T(0)}$. For $x \in D(T)$ define the quantities $\|Tx\|$ and $\|T\|$, respectively by:

$$\|Tx\| := \|QTx\|,$$

$$\|T\| := \|QT\|.$$