Linear Algebra and its Applications 462 (2014) 133–185

Contents lists available at ScienceDirect

Linear Algebra and its Applications

www.elsevier.com/locate/laa

# Local linear dependence seen through duality II

Clément de Seguins Pazzis

*Université de Versailles Saint-Quentin-en-Yvelines, Laboratoire de Mathématiques de Versailles, 45 avenue des Etats-Unis, 78035 Versailles cedex, France*

A R T I C L E   I N F O

*Article history:*
Received 20 July 2014
Accepted 15 August 2014
Available online 3 September 2014
Submitted by P. Semrl

*MSC:*
47L05
15A03
15A30

*Keywords:*
Local linear dependence
Space of operators
Bounded rank
Alternating maps

A B S T R A C T

A vector space $\mathcal{S}$ of linear operators between finite-dimensional vector spaces $U$ and $V$ is called locally linearly dependent (in abbreviated form: LLD) when every vector $x \in U$ is annihilated by a non-zero operator in $\mathcal{S}$. By a duality argument, one sees that studying LLD operator spaces amounts to studying vector spaces of matrices with rank less than the number of columns, or, alternatively, vector spaces of non-injective operators.

In this article, this insight is used to obtain classification results for LLD spaces of small dimension or large essential range (the essential range being the sum of all the ranges of the operators in $\mathcal{S}$). We show that such classification theorems can be obtained by translating into the context of LLD spaces Atkinson's classification of primitive spaces of bounded rank matrices; we also obtain a new classification theorem for such spaces that covers a range of dimensions for the essential range that is roughly twice as large as that in Atkinson's theorem. In particular, we obtain a classification of all 4-dimensional LLD operator spaces for fields with more than 3 elements (beforehand, such a classification was known only for algebraically closed fields and in the context of primitive spaces of matrices of bounded rank).

These results are applied to obtain improved upper bounds for the maximal rank in a minimal LLD operator space.



*E-mail address:* dsp.prof@gmail.com.

http://dx.doi.org/10.1016/j.laa.2014.08.013
0024-3795/

## 1. Introduction

Throughout the paper, all the vector spaces are assumed to be finite-dimensional.

### 1.1. Local linear dependence

Let $U$ and $V$ be finite-dimensional vector spaces over a field $\mathbb{K}$ whose cardinality is denoted by $\#\mathbb{K}$, and $\mathcal{S}$ be a linear subspace of the space $\mathcal{L}(U,V)$ of all linear maps from $U$ to $V$. We say that $\mathcal{S}$ is **locally linearly dependent** (LLD) when every vector $x \in U$ is annihilated by some non-zero operator $f \in \mathcal{S}$. Given a positive integer $c$, we say that $\mathcal{S}$ is $c$-locally linearly dependent ($c$-LLD) when, for every vector $x \in U$, the linear subspace $\{f \in \mathcal{S}\colon\ f(x)=0\}$ has dimension at least $c$.

Alternatively, a family $(f_1,\dots,f_n)$ is called LLD when, for every $x \in U$, the family $(f_1(x),\dots,f_n(x))$ is linearly dependent in $V$. Obviously, this property is satisfied if and only if either $f_1,\dots,f_n$ are linearly dependent in $\mathcal{L}(U,V)$ or $\operatorname{span}(f_1,\dots,f_n)$ is an LLD operator space. Moreover, if some linear subspace $W$ of $V$ contains the image of each $f_i$ and $\dim W < n$, then $f_1,\dots,f_n$ are obviously LLD.

The following example plays a central part in this article: let $\varphi : U \times U \to V$ be an alternating bilinear map (with $U$ non-zero), and assume that $\varphi$ is fully-regular, that is $V = \operatorname{span}\{\varphi(x,y) \mid (x,y) \in U^2\}$ and $\varphi(x,-) \neq 0$ for all non-zero vectors $x \in U$. Then, the linear subspace $\mathcal{S}_\varphi := \{\varphi(x,-) \mid x \in U\}$ of $\mathcal{L}(U,V)$ is LLD as, for every non-zero vector $x \in U$, one has $\varphi(x,x)=0$ with $\varphi(x,-) \neq 0$. We shall say that $\mathcal{S}_\varphi$ is an operator space of the alternating kind. An obvious example is the one of the standard pairing $\varphi : U \times U \to U \wedge U$.

Two operator spaces $\mathcal{S} \subset \mathcal{L}(U,V)$ and $\mathcal{S}' \subset \mathcal{L}(U',V')$ are called **equivalent**, and we write $\mathcal{S} \sim \mathcal{S}'$, when there are two isomorphisms $F : U \xrightarrow{\simeq} U'$ and $G : V' \xrightarrow{\simeq} V$ such that $\mathcal{S} = \{G \circ g \circ F \mid g \in \mathcal{S}'\}$, in which case we have a uniquely defined isomorphism $H : \mathcal{S} \xrightarrow{\simeq} \mathcal{S}'$ such that

$$\forall f \in \mathcal{S}, \quad f = G \circ H(f) \circ F.$$

The corresponding notion for spaces of rectangular matrices is the standard equivalence relation, where two matrix subspaces $\mathcal{M} \subset \mathrm{M}_{m,n}(\mathbb{K})$ and $\mathcal{M}' \subset \mathrm{M}_{p,q}(\mathbb{K})$ are equivalent if and only if $m=p$, $n=q$ and there are non-singular matrices $P \in \mathrm{GL}_m(\mathbb{K})$ and $Q \in \mathrm{GL}_n(\mathbb{K})$ such that $\mathcal{M} = P\mathcal{M}'Q$. Obviously, the operator spaces $\mathcal{S}$ and $\mathcal{S}'$ are equivalent if and only if they are represented (in arbitrary bases of $U$, $V$, $U'$ and $V'$) by equivalent matrix spaces, or, alternatively, if and only if there are choices of bases of $U$, $V$, $U'$ and $V'$ for which the same space of matrices represents both $\mathcal{S}$ and $\mathcal{S}'$.

Note that if $\mathcal{S} \sim \mathcal{S}'$ and $\mathcal{S}$ is $c$-LLD, then $\mathcal{S}'$ is also $c$-LLD. Moreover, if $\mathcal{S}$ is minimal among the $c$-LLD subspaces of $\mathcal{L}(U,V)$ and $\mathcal{S} \sim \mathcal{S}'$, then $\mathcal{S}'$ is minimal among the $c$-LLD subspaces of $\mathcal{L}(U',V')$. By the classification of minimal $c$-LLD operator spaces, we mean their determination up to equivalence.