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Maps completely preserving commutativity and maps completely preserving Jordan zero-product

Li Huang *, Yanxiao Liu

Department of Mathematics, Taiyuan University of Science and Technology, Taiyuan 030024, PR China

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ABSTRACT

Let X, Y be real or complex Banach spaces with infinite dimension, and let $\mathcal{A}$, $\mathcal{B}$ be standard operator algebras on X and Y, respectively. In this paper, we show that every map completely preserving commutativity from $\mathcal{A}$ onto $\mathcal{B}$ is a scalar multiple of either an isomorphism or (in the complex case) a conjugate isomorphism. Every map completely preserving Jordan zero-product from $\mathcal{A}$ onto $\mathcal{B}$ is a scalar multiple of either an isomorphism or (in the complex case) a conjugate isomorphism.

1. Introduction

Let $\mathcal{A}$ and $\mathcal{B}$ be two algebras over the same field $\mathbb{F}$. A map $\Phi : \mathcal{A} \to \mathcal{B}$ preserves commutativity if $\Phi(A)\Phi(B) = \Phi(B)\Phi(A)$ for every pair of commuting elements $A, B \in \mathcal{A}$.

* Corresponding author.
E-mail addresses: huangli19790315@163.com (L. Huang), 1017216707@qq.com (Y. Liu).

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We further say that Φ preserves commutativity in both directions if for every pair $A, B \in \mathcal{A}$, we have

$$AB = BA \quad \Leftrightarrow \quad \Phi(A)\Phi(B) = \Phi(B)\Phi(A).$$

Commutativity is a very important concept in mathematics, and it can be applied to quantum mechanics. In quantum mechanics, commutativity of self-adjoint operators describes compatibility of observables. In 1976, Watkins [21] characterized the bijective linear maps preserving commutativity on $M_n(\mathbb{F})$ ($n \geq 4$). For more relevant results on linear commutativity preservers, we refer to [1,4,2,12,18,13]. Now let us turn to non-linear commutativity preserver problems. Šemrl [20] characterized the injective continuous non-linear commutativity preserving maps on $M_n(\mathbb{C})$. Molnár and Šemrl [15] characterized the bijective non-linear maps of the set of all self-adjoint bounded linear operators on a complex separable Hilbert space H of dimension at least 3, which preserve commutativity in both directions. In [16], Molnár and Timmermann replaced preserving commutativity with preserving the norm of the commutator of operators, and got the more simple form of the maps. Nagy [17] characterized the bijective transformations on the space of density operators which preserve commutativity. The assumption of preserving commutativity can be reformulated as the assumption of preserving zero Lie products, that is, $AB-BA = 0 \Leftrightarrow \Phi(A)\Phi(B)-\Phi(B)\Phi(A) = 0$ for every pair $A, B \in \mathcal{A}$. A related preserver problem is a characterization of maps preserving Jordan zero-product, that is,

$$AB + BA = 0 \quad \Leftrightarrow \quad \Phi(A)\Phi(B) + \Phi(B)\Phi(A) = 0$$

for every pair $A, B \in \mathcal{A}$. In [3], Chebotar et al. described maps preserving Jordan zero-products on the Jordan algebra of all Hermitian operators. In [22], Zhao and Hou characterized additive preservers of zeros of Jordan product on certain operator algebras. In this paper, it is our aim to characterize the general surjective maps between standard operator algebras on Banach spaces that completely preserve commutativity or Jordan zero-product of operators.

As is well-known, completely preserver problems can be more precise in reflecting the homomorphic maps between operator algebras. Let X and Y be Banach spaces over $\mathbb{F}$. Let $\mathcal{B}(X)$ denote the Banach algebra of all bounded linear operators from X into X. Let $\mathcal{S} \subseteq \mathcal{B}(X)$ and $\mathcal{T} \subseteq \mathcal{B}(Y)$ be linear subspaces, and let $\Phi : \mathcal{S} \to \mathcal{T}$ be a map. Define, for each $n \in \mathbb{N}$, a map $\Phi_n : \mathcal{S} \otimes M_n(\mathbb{F}) \to \mathcal{T} \otimes M_n(\mathbb{F})$ by

$$\Phi_n\big((s_{ij})_{n\times n}\big) = \big(\Phi(s_{ij})\big)_{n\times n}.$$

Then Φ is said to be n-commutativity preserving if Φ_n preserves commutativity, Φ is said to be completely commutativity preserving if Φ is n-commutativity preserving for every positive integer n. Similarly, one can introduce the concepts of n-Jordan zero-product preserving maps and completely Jordan zero-product preserving maps. In this respect,