Linear Algebra and its Applications 465 (2015) 15–42

Contents lists available at ScienceDirect

Linear Algebra and its Applications

www.elsevier.com/locate/laa

Matrix characterizations of Riordan arrays

Tian-Xiao He

Department of Mathematics, Illinois Wesleyan University, Bloomington, IL 61702-2900, USA

A R T I C L E   I N F O

Article history:
Received 7 July 2013
Accepted 4 September 2014
Available online 25 September 2014
Submitted by R. Brualdi

MSC:
05A15
05A05
11B39
11B73
15B36
15A06
05A19
11B83

Keywords:
Riordan group
Generating function
Stieltjes matrix
Succession rule
Production matrix
A-matrix
Fundamental theorem of Riordan arrays
Stirling numbers of the second kind

A B S T R A C T

Here we discuss two matrix characterizations of Riordan arrays, P-matrix characterization and A-matrix characterization. P-matrix is an extension of the Stieltjes matrix defined in [28] and the production matrix defined in [8]. By modifying the marked succession rule introduced in [21], a combinatorial interpretation of the P-matrix is given. The P-matrix characterizations of some subgroups of Riordan group are presented, which are used to find some algebraic structures of the subgroups. We also give the P-matrix characterizations of the inverse of a Riordan array and the product of two Riordan arrays. A-matrix characterization is defined in [20], and it is proved to be a useful tool for a Riordan array, while, on the other side, the A-sequence characterization is very complex sometimes. By using the fundamental theorem of Riordan arrays, a method of construction of A-matrix characterizations from Riordan arrays is given. The converse process is also discussed. Several examples and applications of two matrix characterizations are presented.

http://dx.doi.org/10.1016/j.laa.2014.09.008

1. Introduction

Riordan arrays are infinite, lower triangular matrices defined by the generating function of their columns. They form a group, called *the Riordan group* (see Shapiro et al. [29]). Some of the main results on the Riordan group and its application to combinatorial sums and identities can be found in Luzón, Merlini, Morón, and Sprugnoli [18,19] and Sprugnoli [30,31], on subgroups of the Riordan group in Cheon and Jin [3], Jean-Louis and Nkwanta [17], Peart and Woan [23], and Shapiro [26], on some characterizations of Riordan matrices in Rogers [25], Merlini, Rogers, Sprugnoli, and Verri [20], and He and Sprugnoli [15], and on many interesting related results in Cheon, Kim, and Shapiro [4,5], Gould and He [10], He [11–13], He, Hsu, and Shiue [14], Nkwanta [22], Shapiro [27,28], Wang and Wang [34], and so forth.

More formally, let us consider the set of formal power series (f.p.s.) $\mathcal{F} = \mathbb{R}[\![t]\!]$; the *order* of $f(t) \in \mathcal{F}$, $f(t) = \sum_{k=0}^{\infty} f_k t^k$ ($f_k \in \mathbb{R}$), is the minimal number $r \in \mathbb{N}$ such that $f_r \neq 0$; $\mathcal{F}_r$ is the set of formal power series of order r. It is known that $\mathcal{F}_0$ is the set of *invertible* f.p.s. and $\mathcal{F}_1$ is the set of *compositionally invertible* f.p.s., that is, the f.p.s. $f(t)$ for which the compositional inverse $\bar{f}(t)$ exists such that $f(\bar{f}(t)) = \bar{f}(f(t)) = t$. Let $d(t) \in \mathcal{F}_0$ and $h(t) \in \mathcal{F}_1$; the pair $(d(t), h(t))$ defines the *(proper) Riordan array* $D = (d_{n,k})_{n,k\in\mathbb{N}} = (d(t), h(t))$ having

$$d_{n,k} = \left[t^n\right] d(t)h(t)^k \tag{1}$$

or, in other words, having $d(t)h(t)^k$ as the generating function whose coefficients make-up the entries of column k.

It immediately knows that the usual row-by-column product of two Riordan arrays is also a Riordan array:

$$\bigl(d_1(t), h_1(t)\bigr) * \bigl(d_2(t), h_2(t)\bigr) = \bigl(d_1(t)d_2\bigl(h_1(t)\bigr), h_2\bigl(h_1(t)\bigr)\bigr). \tag{2}$$

The Riordan array $I = (1, t)$ is everywhere 0 except that it contains all 1's on the main diagonal; it can be easily proved that I acts as an identity for this product, that is, $(1, t) * (d(t), h(t)) = (d(t), h(t)) * (1, t) = (d(t), h(t))$. From these facts, we deduce a formula for the inverse Riordan array:

$$\bigl(d(t), h(t)\bigr)^{-1} = \left(\frac{1}{d(\bar{h}(t))}, \bar{h}(t)\right) \tag{3}$$

where $\bar{h}(t)$ is the compositional inverse of $h(t)$. In this way, the set $\mathcal{R}$ of proper Riordan arrays forms a group.

Several subgroups of $\mathcal{R}$ are important and have been considered in the literature:

- the *Appell subgroup* is the set $\mathcal{A}$ of the Riordan arrays $D = (d(t), t)$; it is an invariant subgroup and is isomorphic to the group of f.p.s.'s of order 0, with the usual product as group operation;