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Polynomial numerical hulls of the direct sum of a normal matrix and a Jordan block

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ABSTRACT

Let $J_k(\lambda)$ be the $k \times k$ Jordan block with eigenvalue λ and let N be an $m \times m$ normal matrix. In this paper we study the polynomial numerical hulls of order 2 and $n-1$ for $A = J_k(\lambda) \oplus N$, where $n = m + k$. We obtain a necessary and sufficient condition such that $V^2(A)$ has an interior point. Also, we analytically characterize $V^2(J_2(\lambda) \oplus N)$ and we show that if $\sigma(N) \cup \{\lambda\}$ is co-linear, then $V^2(J_2(\lambda) \oplus N) = \bigcup_{a \in \sigma(N)} V^2(J_2(\lambda) \oplus [a])$. Finally, we study $V^{n-1}(A)$ and we show that if $\sigma(N)$ is neither co-linear nor co-circular, then $V^{n-1}(A)$ has at most one point more than $\sigma(A)$.

1. Introduction and preliminaries

Let M_n be the set of all $n \times n$ complex matrices. Motivated by the study of convergence of projection methods in solving linear systems such as GMRES (e.g., see [1,2]), researchers studied the polynomial numerical hulls of order k of a matrix $A \in M_n$, which is defined and denoted by

$$V^k(A) = \{\mu \in \mathbb{C} : |p(\mu)| \le \|p(A)\|, \ \forall p \in \mathcal{P}_k\}, \tag{1}$$

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where $\mathcal{P}_k$ is the set of complex polynomials with degree at most k. The sets $V^k(A)$ are nonempty and compact and $V^1(A)$ is equal to the field of values of A [1]. An equivalent definition of polynomial numerical hull of order k based on joint numerical range (or k-dimensional generalized field of values) has been established in [2] (see also [3,4]):

$$V^k(A) = \{\mu \in \mathbb{C} : (\mu, \mu^2, \ldots, \mu^k) \in \operatorname{Conv}(W(A, A^2, \ldots, A^k))\}, \tag{2}$$

where $\operatorname{Conv}(S)$ denotes the convex hull of the set $S \subseteq \mathbb{C}^k$ and the joint numerical range of the $n \times n$ matrices $A_1, A_2, \ldots, A_k$ is defined by

$$W(A_1, A_2, \ldots, A_k) = \{(x^*A_1x, x^*A_2x, \ldots, x^*A_kx) : x \in \mathbb{C}^n,\ x^*x = 1\}.$$

Also another useful equivalent definition of polynomial numerical hulls of order k which studied in [5] and [6] is the following

$$V^k(A) := \{z \in \mathbb{C} : p(z) \in W(p(A)),\ \forall p \in \mathcal{P}_k\}. \tag{3}$$

Let $\operatorname{pco}_k(S)$ denote the polynomially convex hull of degree k for a compact set $S \subseteq \mathbb{C}$:

$$\operatorname{pco}_k(S) := \Big\{z \in \mathbb{C} : |p(z)| \le \max_{w \in S} |p(w)|,\ \forall p \in \mathcal{P}_k\Big\}. \tag{4}$$

A set S is called polynomially convex of degree k if $\operatorname{pco}_k(S) = S$.

Lemma 1.1. *(See [3,7].) Let $A \in M_n$. Then the following properties hold for polynomial numerical hulls of order k:*

1. $\sigma(A) = V^n(A) \subseteq V^{k+1}(A) \subseteq V^k(A) \subseteq V^1(A) = W(A)$ *for all* $k = 2, \ldots, n-1$.
2. $V^k(A)$ *is polynomially convex of degree k and* $\operatorname{pco}_k(\partial V^k(A)) = V^k(A)$.
3. *We have* $V^k(U^*AU) = V^k(A)$ *for any unitary matrix* $U \in M_n$ *and for all* $k = 1, \ldots, n$.
4. $V^k(\alpha A + \beta I) = \alpha V^k(A) + \beta$ *for all α and β in the complex plane* $\mathbb{C}$.
5. *Let A be a Hermitian matrix. Then* $V^2(A) = \sigma(A)$.

In [8], it has been shown that the polynomial numerical hulls of order $k < n$ for a Jordan block $J_n(\lambda)$ is a disk about λ and the radius of the disk has been obtained when $k = n-1$. Also in [7] the polynomial numerical hulls of normal matrices has been studied, specially in [9] the polynomial numerical hull of order two for all normal matrices and matrices whose squares are Hermitian have been characterized.

While polynomial numerical hulls appear to be a valuable tool, their determination or computation represents a difficult open problem even for simple classes of nonnormal matrices [10]. In this paper, by using the results in [2,8] and [9], we study polynomial numerical hulls of particular nonnormal matrices. More precisely we study $V^t(J_k(\lambda) \oplus N)$