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Operator inequalities: From a general theorem to concrete inequalities

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ABSTRACT

The aim of this paper is to give a method to extract concrete inequalities from a general theorem, which is established by making use of majorization relation between functions. By this method we can get a lot of inequalities; among others we extend Furuta inequality as follows: Let f_i, g_j be positive operator monotone functions on $[0, \infty)$ and put $k(t) = t^{r_0} f_1(t)^{r_1} \cdots f_m(t)^{r_m}$, $h(t) = t^{p_0} g_1(t)^{p_1} \cdots g_n(t)^{p_n}$, where $p_0 \geq 1$ and $r_i \geq 0$, $p_j \geq 0$. Then $0 \leq A \leq C \leq B$ implies, for $0 < \alpha \leq \frac{1+r_0}{p+r_0}$ with $p = p_0 + \cdots + p_n$, $(k(C)^{\frac{1}{2}} h(A) k(C)^{\frac{1}{2}})^\alpha \leq (k(C)^{\frac{1}{2}} h(C) k(C)^{\frac{1}{2}})^\alpha \leq (k(C)^{\frac{1}{2}} h(B) k(C)^{\frac{1}{2}})^\alpha$. Moreover, we show $\log C^{1/2} e^A C^{1/2} \leq \log C^{1/2} e^C C^{1/2} \leq \log C^{1/2} e^B C^{1/2}$, provided C is invertible. We also refer to operator geometric mean.

1. Introduction

Let $\mathbf{P}(I)$ denote the set of all operator monotone functions on an interval I. A constant function is here admitted to be an operator monotone function. We put $\mathbf{P}_+(I) = \{f \in$

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$\mathbf{P}(I) \mid f(t) \geq 0$ for $t \in I,\ f \neq 0\}$. It is evident and fundamental that $-\frac{1}{t} \in \mathbf{P}(-\infty, 0) \cap \mathbf{P}(0, \infty)$. If $f \in \mathbf{P}_+(a, b)$ and $-\infty < a$, then f has the natural extension to $[a, b)$, which belongs to $\mathbf{P}_+[a, b)$. We therefore identify $\mathbf{P}_+(a, b)$ with $\mathbf{P}_+[a, b)$. It is well-known that if $f(t) \in \mathbf{P}_+(0, \infty)$, then $\frac{t}{f(t)}$ and $f(t^\alpha)^{1/\alpha}$ $(0 < \alpha < 1)$ are both in $\mathbf{P}_+(0, \infty)$ and that if $f(t), \phi(t)$ and $\varphi(t)$ are all in $\mathbf{P}_+(0, \infty)$, then so are $f(t^\alpha)\phi(t^{1-\alpha})$, $f(t)^\alpha \phi(t)^{1-\alpha}$ for $0 < \alpha < 1$ and $\phi(f(t))\varphi(\frac{t}{f(t)})$. This result deeply depends on Loewner's theorem ([16], also see [15,5,8,13,18,19]).

By Krauss [14] and Bendat–Shermann [4], $g(t)$ defined on $[0, \infty)$ with $g(0) = 0$ is operator convex if and only if $g(t) = tf(t)$ with $f(t) \in \mathbf{P}(0, \infty)$. It is also known that a function $f(t)$ defined on (a, ∞) is operator monotone if and only if $f(t)$ is operator concave and $f(\infty) > -\infty$ [26]. See [6,7,12,27,28] about recent study around this area.

Hansen [10] and Hansen–Pedersen [11] have shown that if $f(t) \in \mathbf{P}_+(0, \infty)$, $\|X\| \leq 1$ and $A \geq 0$, then $X^* f(A) X \leq f(X^* A X)$.

From now on, we assume that a function is continuous and that 'increasing' means 'strictly increasing'. We also assume that $I = [a, b)$ or $I = (a, b)$ with $-\infty \leq a < b \leq \infty$ unless otherwise stated.

Definition 1.1. (See [24,25].) Let $h(t)$ and $g(t)$ be functions defined on I, and suppose $g(t)$ is increasing. Then h is said to be *majorized* by g, in symbol $h \preceq g$ if the composite $h \circ g^{-1}$ is operator monotone on $g(I)$.

This definition is equivalent to

$$\sigma(A), \sigma(B) \subset I, \qquad g(A) \leq g(B) \quad \Longrightarrow \quad h(A) \leq h(B).$$

In this case $h(t)$ is clearly non-decreasing. If we need to make clear the domain I, we write $h \preceq g(I)$ or $h \preceq g$ on I. $f(t) \preceq t$ on I means $f \in \mathbf{P}(I)$. The Loewner–Heinz inequality says $g(t)^\alpha \preceq g(t)^\beta$ if $0 < \alpha < \beta$ and $g(t) > 0$ is increasing. Let τ be an increasing function from an interval J to I. Then $h \circ \tau \preceq g \circ \tau$ on J if $h \preceq g$ on I. The following fact will be used later: *Let* $h(t) \geq 0$ *be a non-decreasing function on* $(0, \infty)$ *and* $g(t)$ *an increasing function on* $(0, \infty)$ *with the range* $(0, \infty)$. *Then for* $0 < \alpha < 1$

$$h(t) \preceq g(t) \text{ on } (0, \infty) \quad \Longrightarrow \quad h(t)^{1/\alpha} \preceq g(t)^{1/\alpha} \text{ on } (0, \infty).$$

Indeed, the hypothesis implies $f(s) := h(g^{-1}(s)) \in \mathbf{P}_+(0, \infty)$. Since $(h(g^{-1}(s^\alpha)))^{1/\alpha} = f(s^\alpha)^{1/\alpha} \preceq s$ on $0 < s < \infty$, by putting $s = g(t)^{1/\alpha}$ we get $h(t)^{1/\alpha} \preceq g(t)^{1/\alpha}$ on $0 < t < \infty$. The following set was introduced in [24,25]

$$\mathbf{LP}_+(I) := \{h \mid h \text{ is defined on } I,\ h(t) > 0 \text{ on } (a, b),\ \log h \in \mathbf{P}(a, b)\}.$$
$$\mathbf{P}_+^{-1}(I) := \{h \mid h \text{ is increasing on } I,\ h\big((a, b)\big) = (0, \infty), h^{-1} \in \mathbf{P}\big(h(I)\big)\}.$$

If $-\infty < a$, identifying h on (a, b) as its natural extension to $[a, b)$ gives $\mathbf{LP}_+(a, b) = \mathbf{LP}_+[a, b)$, $\mathbf{P}_+^{-1}(a, b) = \mathbf{P}_+^{-1}[a, b)$. Notice that $h \in \mathbf{P}_+^{-1}(a, b)$ if and only if $t \preceq h(t)$ on (a, b) and $h(a, b) = (0, \infty)$.