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Notes on the second largest eigenvalue of a graph ☆

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For a fixed real number r we give several necessary and/or sufficient conditions for a graph to have the second largest eigenvalue of the adjacency matrix, or signless Laplacian matrix, less then or equal to r.

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Signless Laplacian
Second largest eigenvalue

1. Introduction

Let $G = (V(G), E(G))$ be a simple graph of order $n = |V(G)|$ and size $m = |E(G)|$. If an edge $e \in E(G)$ joins vertices $u, v \in V(G)$ then, for short, uv stands for e. The adjacency matrix of G is denoted by $A(G)$, while $P(x; G) = \det(xI - A(G))$ is the characteristic polynomial of G. Roots of $P(x; G)$ comprise the *spectrum* of G, denoted by $\mathtt{Sp}(G)$ (note, it is real since $A(G)$ is symmetric). Let $\lambda_1(G) \geq \lambda_2(G) \geq \cdots \geq \lambda_n(G)$ be the eigenvalues of G given in non-increasing order. Recall, $\lambda_1(G) > \lambda_2(G)$ if G is connected (in sequel, if not told otherwise, we will consider only connected graphs). In particular, $\lambda_1(G)$ is called the *index* of G. For a given $\lambda \in \mathrm{Sp}(G)$, $m(\lambda; G)$ denotes its *multiplicity* (note, since $A(G)$ is symmetric, the algebraic and geometric multiplicities of λ are equal). The eigenvalues of multiplicity one are called the *simple eigenvalues*.

The equation $A\mathbf{x} = \lambda\mathbf{x}$ is called the *eigenvalue equation* of A, or of a labeled graph G if $A = A(G)$. For a fixed $\lambda \in \mathrm{Sp}(G)$, its non-trivial solution $\mathbf{x} = (x_1, x_2, \ldots, x_n)^T$ is a λ-eigenvector of A, or of a labeled graph G. In particular, if $\lambda = \lambda_1(G)$ then the corresponding vector (with positive coordinates) is called a *principal eigenvector* of G.

In the scalar form, for any $\lambda \in \mathrm{Sp}(G)$, the eigenvalue equation reads:

$$\lambda x_u = \sum_{v \sim u} x_v,$$

where $u \in V(G)$. The null space of $A(G) - \lambda I$ is called the eigenspace of G and is denoted by $\mathcal{E}(\lambda; G)$. Note also that each eigenvector can be interpreted as a mapping $\mathbf{x} : V(G) \to \mathbb{R}$. So if $v \in V(G)$, $\mathbf{x}(v)$ and x_v can be identified.

The spectral decomposition of any symmetric matrix A, or of graph G if $A = A(G)$, is given by $A = \mu_1 P_1 + \mu_2 P_2 + \cdots + \mu_\ell P_\ell$, where μ_i's are the distinct eigenvalues of A. For $i = 1, 2, \ldots, \ell$, P_i is a projection matrix which maps the whole space $\mathbb{R}^n$ onto $\mathcal{E}(\mu_i; G)$. The quantity $\alpha_{i,v} = |P_i \mathbf{e}_v|$ is called the *graph angle* between $\mathbf{e}_v$ and $\mathcal{E}(\mu_i)$ (or, more precisely, it is just the cosine of that angle).

We denote by $G - u$ ($G - U$) the subgraph of G obtained by deleting a vertex u (resp. a vertex set U) from G. If $U \subseteq V(G)$ then $\langle U \rangle$ denotes the induced subgraph of G. If H is an induced subgraph of G, then we write $H \subseteq G$ (or $H \subset G$ if H is a proper subgraph of G). If $H \subset G$, and if $\{u\} \cap V(H) = \emptyset$ (or, more generally, $U \cap V(H) = \emptyset$) then $H + u = \langle V(H) \cup \{u\} \rangle$ (resp. $H + U = \langle V(H) \cup U \rangle$).

Recall, in general, if any vertex is deleted from some graph, then the multiplicity of any eigenvalue changes at most by one (by the Interlacing theorem; see, for example, [2, p. 17]). A vertex v of G is called the *downer* (*neutral*, *Parter*) vertex for μ_i if $m(\mu_i; G - v)$ is equal to $m(\mu_i; G) - 1$ (resp. $m(\mu_i; G)$, $m(\mu_i; G) + 1$). Note, if vertex v is a downer for μ_i then $\alpha_{i,v} \neq 0$ (for more details, see [4]).