Linear Algebra and its Applications 465 (2015) 325–346

Contents lists available at ScienceDirect

Linear Algebra and its Applications

www.elsevier.com/locate/laa

The power mean and the least squares mean of probability measures on the space of positive definite matrices

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ARTICLE INFO

Article history:
Received 21 March 2014
Accepted 27 September 2014
Available online 10 October 2014
Submitted by R. Bhatia

MSC:
60B12
8A25
15B48

Keywords:
Bochner integral
Compactly supported measure
Power mean
Least squares mean

ABSTRACT

In this paper we derive properties of the least squares (or Karcher) mean of probability measures on the open cone Ω of positive definite matrices of some fixed dimension endowed with the trace metric that generalize known properties of the weighted least squares mean of finitely many positive definite matrices. Our approach is based on first defining the t-power mean of a probability measure as the unique fixed point of the contractive map

$$X \in \Omega \mapsto \int_{\Omega} X \#_t Z \mu(dZ)$$

with respect to the Thompson metric, establishing its properties analogous to those of the power mean for a finite number of positive definite matrices, and showing the t-power means converge to the Karcher mean as $t \to 0$. We carry out this program first of all for the compactly supported probability measures and show that theory including the monotonicity extends to the general case.

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http://dx.doi.org/10.1016/j.laa.2014.09.042

1. Introduction

Let Ω be the open convex cone of $m \times m$ positive definite (Hermitian) matrices. Recall that the Riemannian trace distance between A and B in Ω is given by

$$d(A,B) := \left[\sum_{j=1}^{m} \log^2 \lambda_j\left(A^{-1}B\right)\right]^{1/2},$$

where $\lambda_j(X)$ denotes the j-th eigenvalue of X in ascending order. For any $t \in (0,1]$ Y. Lim and M. Pálfia have defined in [10] a t-power mean $P_t(\omega, \mathbb{A})$ of n-tuple $\mathbb{A} = (A_1, \ldots, A_n) \in \Omega^n$ with positive probability vector $\omega = (w_1, \ldots, w_n)$ as the unique positive definite solution X of the equation

$$X = \sum_{i=1}^{n} w_i X \#_t A_i, \tag{1.1}$$

where $A \#_t B := A^{1/2}(A^{-1/2}BA^{-1/2})^t A^{1/2}$ is the weighted geometric mean of A and B. Furthermore, they have proved that the limit of t-power means as $t \to 0$ is the least squares mean $\Lambda(\omega, \mathbb{A})$, also called the Karcher mean, which is the unique minimizer of the sum of squares of the Riemannian trace distances to each of the A_i:

$$\Lambda(\omega, \mathbb{A}) = \underset{X \in \Omega}{\arg\min} \sum_{i=1}^{n} w_i d^2(X, A_i). \tag{1.2}$$

Karcher's formula for the gradient of the objective function (Theorem 2.1 of [6]) yields that the least squares mean is the unique positive definite solution of the equation

$$\sum_{i=1}^{n} w_i \log\left(X^{-1/2} A_i X^{-1/2}\right) = O. \tag{1.3}$$

Using this idea they also proved the monotonicity of the least squares mean:

$$\Lambda(\omega, \mathbb{A}) \le \Lambda(\omega, \mathbb{B}) \quad \text{whenever } A_i \le B_i \text{ for all } i = 1, \ldots, n.$$

Various numerical methods to find the least squares mean, for instance, fixed point methods, optimization algorithms, and iterative methods have been introduced; see [1] and references therein. In what follows we generalize to a theory of power means and the least squares mean in the setting of probability measures on Ω.

To define power means for probability measures on Ω, we use the vector-valued Bochner integral; some of its well-known properties are reviewed in Section 2. We first consider the probability measure on Ω of compact support and recall in Section 3 useful properties such as the change of variables for pushforward measures. We then define in